Sada příkladů 9/11

Fourierova transformace

Spočtěte Fourierovu transformaci následujících funkcí

1. $f(x) = 1$ pro $x \in [-1, 1]$, $f(x) = 0$ jinak
2. $f(x) = x\mathrm{e}^{-ax^2}$, $x \in \mathbb{R}^N$, $a > 0$
3. $f(x) = \mathrm{e}^{-a|x|^2}$, $x \in \mathbb{R}^N$, $a > 0$
4. $f(x) = \dfrac{1}{a^2 + x^2}$, $a > 0$
5. $f(x) = \dfrac{x}{a^2 + x^2}$, $a > 0$
6. $f(x) = \mathrm{e}^{-a|x|}$, $a > 0$, $x \in \mathbb{R}$
7. $f(x) = \cos bx\, \mathrm{e}^{-a|x|}$, $a > 0$, $b \in \mathbb{R}$, $x \in \mathbb{R}$
8. $f(x) = \dfrac{\sinh ax}{\sinh \pi x}$, $|a| < \pi$
9. $f(x) = \dfrac{1}{|x|^s}\mathrm{e}^{-a|x|}$, $a > 0$, $0 < s < 1$, $x \in \mathbb{R}$
10. $f(x) = \dfrac{1}{(x-\lambda)^m}$, $\lambda \in \mathbb{C}$, $\operatorname{Im} \lambda \neq 0$, $m \in \mathbb{N}$
11. Pomocí Fourierovy transformace vyřešte počáteční úlohu pro rovnici tepla
$$u_t - \Delta u = 0 \text{ v } \mathbb{R}^+ \times \mathbb{R}^N,$$
kde $u(0, x) = u_0$ v $\mathbb{R}^N$.
12. Pomocí Fourierovy transformace vyřešte Cauchyovu úlohu pro obyčejnou diferenciální rovnici
$$-y'' + a^2 y = f \text{ v } \mathbb{R}.$$

FOURIEROVA TRANSFORMACE

Schwartzův prostor $S(\mathbb{R}^N) = \{ f \in C^\infty(\mathbb{R}^N) : \| x^\alpha D^\beta f \|_{L^\infty} < \infty \ \forall \alpha, \beta \in \mathbb{N}_0^N \}$

$D^\beta f := \dfrac{\partial^{\beta_1 + \dots + \beta_N} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \dots \partial x_N^{\beta_N}}$ $\qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}$

Konvoluce: $(f * g)(x) = \int_{\mathbb{R}^N} f(x-y) g(y) \, dy$

Fourierova transformace: $f \in S(\mathbb{R}^N)$. Pak $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot \xi} dx \ \in S(\mathbb{R}^N)$

Zpětná Fourier. transformace : $\mathcal{F}^{-1}(f)(\xi) := \int_{\mathbb{R}^N} f(x) e^{2\pi i x \cdot \xi} dx \ \in S(\mathbb{R}^N)$

Základní vlastnosti:

- $\mathcal{F}^{-1}(f)(\xi) = \mathcal{F}(f)(-\xi)$, $\overline{\mathcal{F}^{-1}(f)(\xi)} = \mathcal{F}(\bar{f})(\xi)$, $\overline{\mathcal{F}(f)(\xi)} = \mathcal{F}^{-1}(\bar{f})(\xi)$
- $\mathcal{F}(f(x-z)) = e^{-2\pi i z \cdot \xi} \mathcal{F}(f)(\xi) \quad \forall z \in \mathbb{R}^N$
- $\mathcal{F}(f)(\xi - \eta) = \mathcal{F}(e^{2\pi i x \cdot \eta} f(x))(\xi) \quad \forall \eta \in \mathbb{R}^N$
- $\mathcal{F}(f(\varepsilon x))(\xi) = |\varepsilon|^{-N} \mathcal{F}(f(x))\left(\frac{\xi}{\varepsilon}\right) \quad \forall \varepsilon \neq 0$
- f sudá / lichá v x_j $\Rightarrow$ $\mathcal{F}(f)$ sudá / lichá v ξ_j
- $\mathcal{F}(D^\alpha f)(\xi) = (2\pi i \xi)^\alpha \mathcal{F}(f)(\xi)$, $\mathcal{F}^{-1}(D^\alpha f)(\xi) = (-2\pi i \xi)^\alpha \mathcal{F}^{-1}(f)(\xi)$
- $D^\alpha(\mathcal{F}(f))(\xi) = \mathcal{F}((-2\pi i x)^\alpha f(x))(\xi)$, $D^\alpha(\mathcal{F}^{-1}(f))(\xi) = \mathcal{F}^{-1}((2\pi i x)^\alpha f(x))(\xi)$
- $\int_{\mathbb{R}^N} f \mathcal{F}(g) \, dx = \int_{\mathbb{R}^N} \mathcal{F}(f) g \, dx$, totéž pro $\mathcal{F}^{-1}$
- $\mathcal{F}(f * g)(\xi) = \mathcal{F}(f)(\xi) \, \mathcal{F}(g)(\xi)$, totéž pro $\mathcal{F}^{-1}$

Rozšíření F.T. mimo $S(\mathbb{R}^N)$: $\underline{\underline{f \in L^1}} \Rightarrow \mathcal{F}(f) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (totéž $\mathcal{F}^{-1}$)

Pokud je $f \in L^1$ a $\mathcal{F}(f) \in L^1$, pak $\mathcal{F}^{-1}(\mathcal{F}(f)) = f$

$\underline{\underline{f \in L^2}}$: $\mathcal{F}(f) = \lim_{R \to \infty} \int_{B_R(0)} f(x) e^{-2\pi i x \cdot \xi} dx$ $\qquad f \in L^2 \Rightarrow \mathcal{F}(f) \in L^2$ a platí

$$\|f\|_{L^2(\mathbb{R}^N)} = \|\mathcal{F}(f)\|_{L^2(\mathbb{R}^N)} = \|\mathcal{F}^{-1}(f)\|_{L^2(\mathbb{R}^N)}$$

(Plancherelova rovnost)

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1) $f(x) = \chi_{[-1,1]}(x) = 1$ pro $x \in [-1,1]$, $= 0$ pro $x \in \mathbb{R} \setminus [-1,1]$ $\qquad f \in L^1 \Rightarrow$ ex. F.T. $\qquad N = 1$

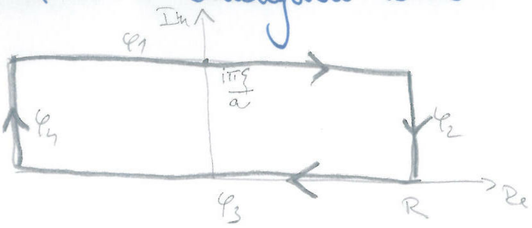
$$\mathcal{F}(f)(\xi) = \int_{-1}^{1} e^{-2\pi i x \cdot \xi} dx = \left[\frac{e^{-2\pi i x \cdot \xi}}{-2\pi i \xi} \right]_{-1}^{1} = \frac{e^{2\pi i \xi} - e^{-2\pi i \xi}}{2\pi i \xi} = \frac{\sin(2\pi \xi)}{\pi \xi}.$$

Přitom $\mathcal{F}(f)(\xi) \notin L^1$, protože $\int_0^\infty \frac{|\sin 2\pi \xi|}{\pi \xi} d\xi = +\infty$ $\quad \left(\|\mathcal{F}(f)\|_{L^1} = 2 \int_0^\infty \frac{|\sin 2\pi \xi|}{\pi \xi} \text{ ze sudosti} \right)$

3) Začneme v 1D: $f(x)=e^{-ax^2}$, $a>0$, $x\in\mathbb{R}$. $f\in S(\mathbb{R})$

$$\mathcal{F}(e^{-ax^2})(\xi)=\int_{-\infty}^{\infty} e^{-ax^2}e^{-2\pi i x\xi}\,dx=\int_{-\infty}^{\infty} e^{-a\left(x^2+\frac{2\pi i x\xi}{a}-\frac{\pi^2\xi^2}{a^2}\right)}\cdot e^{-a\frac{\pi^2\xi^2}{a^2}}\,dx=e^{-\frac{\pi^2\xi^2}{a}}\underbrace{\int_{-\infty}^{\infty}e^{-a\left(x+\frac{\pi i\xi}{a}\right)^2}dx}_{=:I}$$

I spočítáme Cauchyovou větou $\tilde{f}(z)=e^{-az^2}$



$$\int_{\varphi_1}\tilde{f}\to I$$

$$\left|\int_{\varphi_2}\tilde{f}\right|\le\int_0^{\frac{\pi\xi}{a}}e^{-aR^2}\cdot e^{at^2}\,dt\le\frac{\pi\xi}{a}e^{a\frac{\pi^2\xi^2}{a^2}}\cdot e^{-aR^2}\xrightarrow[R\to\infty]{}0$$

$\left|\int_{\varphi_4}\tilde{f}\right|$ stejně

$$\int_{\varphi_3}\tilde{f}=-\int_{-R}^{R}e^{-ax^2}dx\to-\sqrt{\frac{\pi}{a}}$$, to je standardní integrál

$\tilde{f}$ holomorfní $\Rightarrow I=\sqrt{\frac{\pi}{a}}$ a $\underline{\mathcal{F}(e^{-ax^2})(\xi)=\sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2\xi^2}{a}}}$. Pozorování: $a=\pi$:
$\mathcal{F}(e^{-\pi x^2})(\xi)=e^{-\pi\xi^2}$

Ve více dimenzích: $$\mathcal{F}(e^{-a|x|^2})(\xi)=\int_{\mathbb{R}^N}e^{-a\sum x_j^2}e^{-2\pi i\sum x_j\xi_j}\,dx=\prod_{j=1}^{N}\int_{-\infty}^{\infty}e^{-ax_j^2}e^{-2\pi i x_j\xi_j}\,dx_j=\underline{\left(\frac{\pi}{a}\right)^{\frac{N}{2}}e^{-\frac{\pi^2|\xi|^2}{a}}}$$

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2) $f(x)=xe^{-ax^2}$, $x\in\mathbb{R}$, $a>0$.   Všimneme si $(e^{-ax^2})'=-2ax\,e^{-ax^2}$
$f\in S(\mathbb{R})$ $\Rightarrow xe^{-ax^2}=-\frac{1}{2a}(e^{-ax^2})'$

$$\mathcal{F}(xe^{-ax^2})(\xi)=-\frac{1}{2a}\left(\mathcal{F}\left((e^{-ax^2})'\right)\right)(\xi)=-\frac{1}{2a}\cdot\left(2\pi i\xi\,\mathcal{F}(e^{-ax^2})(\xi)\right)=\underline{-\frac{\pi i\xi}{a}\cdot\sqrt{\frac{\pi}{a}}e^{-\frac{\pi^2\xi^2}{a}}}$$

Ve více dimenzích dává smysl $f(x)=x_je^{-a|x|^2}$ pro $j\in\{1,\dots,N\}$ a vyjde $-\frac{\pi i\xi_j}{a}\cdot\left(\frac{\pi}{a}\right)^{\frac{N}{2}}e^{-\frac{\pi^2|\xi|^2}{a}}$

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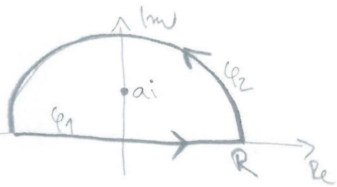
4) $f(x)=\frac{1}{a^2+x^2}$, $a>0$. $f\notin S$, ale $f\in L^1$.

$$\mathcal{F}(f)(\xi)=\int_{-\infty}^{\infty}\underbrace{\frac{e^{-2\pi i x\xi}}{a^2+x^2}}_{g}dx$$ f sudá $\Rightarrow\mathcal{F}(f)$ sudá. Stačí počítat pro $\xi\le0$.

Reziduovou větou: $\int_{\varphi_1}g(z)\,dz\to I$ $\int_{\varphi_2}g(z)$: Jordanovo lemma: $\alpha=-2\pi\xi\ge0$ (proto horní polorovina)



$M_R\sim\frac{1}{R^2}$, to je OK i pro $\alpha=0$. $\Rightarrow\int_{\varphi_2}g\to0$

$$\Rightarrow I=2\pi i\,\mathrm{Res}_{ai}\,g=\frac{e^{-2\pi i\,ai\,\xi}}{2ai}2\pi i=\frac{e^{2\pi a\xi}\cdot\pi}{a}$$

Pro $\xi>0$ musíme prodloužit sudě: $\underline{\mathcal{F}(f)(\xi)=\frac{\pi}{a}e^{-2\pi a|\xi|}}$

5) DÚ

③

6) $f(x) = e^{-a|x|}$, $a>0$, $x\in\mathbb{R}$. $f\notin S(\mathbb{R})$, f sudá, kvůli hladkosti v 0, $f\in L^1$

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-2\pi i x\xi}\,dx = \int_0^{\infty} e^{x\cdot(-a-2\pi i\xi)}\,dx + \int_{-\infty}^{0} e^{x(a-2\pi i\xi)}\,dx$$

$$= \left[\frac{e^{x(-a-2\pi i\xi)}}{-a-2\pi i\xi}\right]_0^{\infty} + \left[\frac{e^{x(a-2\pi i\xi)}}{a-2\pi i\xi}\right]_{-\infty}^{0} = \frac{1}{a+2\pi i\xi} + \frac{1}{a-2\pi i\xi} = \underline{\underline{\frac{2a}{a^2+4\pi^2\xi^2}}}$$

Členy u nekonečna zmizí, protože $e^{-ax}\to 0$ pro $x\to\infty$

a $|e^{2\pi i x\xi}| = 1$. Srovnejte s příkladem 4)

7) $f(x) = \cos bx\, e^{-a|x|}$ $a>0$, $b\in\mathbb{R}$, $x\in\mathbb{R}$. $f\in L^1$, f sudá. Použijeme $\cos bx = \dfrac{e^{ibx}+e^{-ibx}}{2}$

a podobně jako výše dostaneme $\mathcal{F}(f)(\xi) = \frac12\int_{-\infty}^{0} e^{x\cdot(a+ib-2\pi i\xi)}\,dx + \frac12\int_{-\infty}^{0} e^{x(a-ib-2\pi i\xi)}\,dx + \frac12\int_0^{\infty} e^{x(-a+ib-2\pi i\xi)}\,dx + \frac12\int_0^{\infty} e^{x(-a-ib-2\pi i\xi)}\,dx$

Teď stačí jen spočítat $\frac12\cdot\left[\frac{1}{a+ib+2\pi i\xi} + \frac{1}{a-ib+2\pi i\xi} + \frac{1}{a+ib-2\pi i\xi} + \frac{1}{a-ib-2\pi i\xi}\right] = \dots$

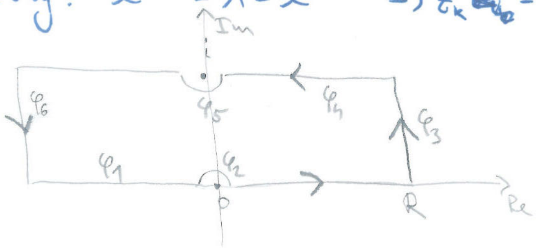
$$\dots = \frac{2a(a^2+b^2+4\pi^2\xi^2)}{(a^2+b^2-4b\pi\xi+4\pi^2\xi^2)(a^2+b^2+4b\pi\xi+4\pi^2\xi^2)} = 2a\cdot\left(\frac{1}{a^2+(b-2\pi\xi)^2} + \frac{1}{a^2+(b+2\pi\xi)^2}\right)$$

8) $f(x) = \dfrac{\sinh ax}{\sinh \pi x}$, $|a|<\pi$ f sudá, $f\in S(\mathbb{R})$

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} \frac{e^{ax}-e^{-ax}}{e^{\pi x}-e^{-\pi x}} e^{-2\pi i x\xi}\,dx = \underbrace{\int_{-\infty}^{\infty} \frac{e^{x(a-2\pi i\xi)}}{e^{\pi x}-e^{-\pi x}}\,dx}_{I_1} + \underbrace{-\int_{-\infty}^{\infty} \frac{e^{x(-a-2\pi i\xi)}}{e^{\pi x}-e^{-\pi x}}\,dx}_{I_2}$$

Póly: $e^{2\pi z} = 1 = e^{2k\pi i} \Rightarrow z_k = ki$

$g_1(z) = \dfrac{e^{z(a-2\pi i\xi)}}{e^{\pi z}-e^{-\pi z}}$



$\int_{\varphi_1} g_1(z)\,dz \to I_1$

$\int_{\varphi_2} g_1(z)\,dz \to iAb$, $b=-\pi$, $A = \lim_{z\to 0} \dfrac{e^{z(a-2\pi i\xi)}\, z}{e^{\pi z}-e^{-\pi z}} = \dfrac{1}{2\pi} \Rightarrow iAb = -\frac12 i$

$\int_{\varphi_3} g_1(z) \to 0$, protože $\left|\int_{\varphi_3} g_1(z)\right| \leq \dfrac{e^{aR}\cdot C}{e^{\pi R}}$ a $a<\pi$

$\int_{\varphi_4} g_1(z)\,dz \to e^{2\pi\xi+ia}\int_{\infty}^{-\infty} \dfrac{e^{z(a-2\pi i\xi)}}{\underbrace{e^{\pi i}}_{=-1} e^{\pi z} - \underbrace{e^{-\pi i}}_{=-1} e^{-\pi z}}\,dz = e^{2\pi\xi+ia}\, I_1$

$\int_{\varphi_5} g_1(z)\,dz \to iAb$, $b=-\pi$, $A = \lim_{z\to i} \dfrac{e^{z(a-2\pi i\xi)}(z-i)}{e^{\pi z}-e^{-\pi z}} = e^{2\pi\xi+ia}\cdot\left(-\dfrac{1}{2\pi}\right) \Rightarrow iAb = \dfrac{e^{2\pi\xi+ia}\, i}{2}$

$\int_{\varphi_6} g_1(z)\,dz \to 0$ podobně jako $\int_{\varphi_3}$. Celkem Cauchyho věta $I_1\cdot\left(1+e^{2\pi\xi+ia}\right) + \frac{i}{2}\left(-1+e^{2\pi\xi+ia}\right) = 0$

$$I_1 = \frac{i}{2}\,\frac{1-e^{2\pi\xi+ia}}{1+e^{2\pi\xi+ia}}$$

I_2 úplně stejně jako I_1, jen všude bude $-a$ místo a.

Integrály přes φ_3 a φ_6 opět zmizí, protože $|a|<\pi$.

$$\Rightarrow \mathcal{F}(f)(\xi) = \frac{i}{2}\cdot\left(\frac{1-e^{2\pi\xi+ia}}{1+e^{2\pi\xi+ia}} - \frac{1-e^{2\pi\xi-ia}}{1+e^{2\pi\xi-ia}}\right) = \frac{i}{2}\,\frac{2e^{2\pi\xi-ia}-2e^{2\pi\xi+ia}}{1+e^{2\pi\xi+ia}+e^{2\pi\xi-ia}+e^{4\pi\xi}} =$$

$$= i\,\frac{e^{-ia}-e^{ia}}{e^{-2\pi\xi}+e^{2\pi\xi}+e^{ia}+e^{-ia}} = \underline{\underline{\frac{\sin a}{\cos a+\cosh(2\pi\xi)}}}$$

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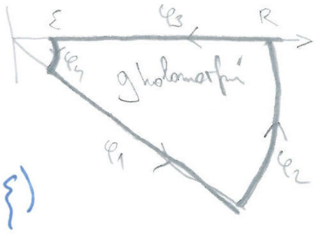
9) $f(x)=\frac{1}{|x|^s}e^{-a|x|}$, $a>0$, $s\in(0,1)$, $x\in\mathbb{R}$, $f\in L^1$, f sudá $\Rightarrow \mathcal{F}(f)$ sudá. BÚNO $\xi>0$.

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty}\frac{1}{|x|^s}e^{-a|x|}\cdot e^{-2\pi i x\xi}\,dx = \int_0^\infty \frac{1}{t^s}e^{-at}e^{2\pi i t\xi}\,dt + \int_0^\infty \frac{1}{t^s}e^{-at}e^{-2\pi i t\xi}\,dt = I^- + I^+$$

Ozn. $g(z)=z^{-s}e^{-z}$. Budeme integrovat tuto fci přes následující křivky, kde $\xi>0$.

I^-:



$\varphi_1(t)=t\cdot(a-2\pi i\xi)$

$\varphi_1'(t)=a-2\pi i\xi$

$$\int_{\varphi_1} g(z)\,dz = (a-2\pi i\xi)\int_\varepsilon^R \frac{1}{t^s}\cdot\frac{1}{(a-2\pi i\xi)^s}e^{-at}e^{2\pi i\xi t}\,dt$$

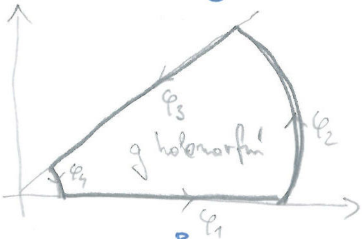
$$\to (a-2\pi i\xi)^{1-s}\, I^-$$

$\int_{\varphi_2} g(z)\,dz \to 0$ Jordanem, $M_R\sim R^{-s}e^{-R}$, $RM_R\to 0$

$$\int_{\varphi_3} g(z)\,dz = -\int_\varepsilon^R \frac{1}{t^s}e^{-t}\,dt \to -\Gamma(1-s)$$

$$\left|\int_{\varphi_4} g(z)\,dz\right| \le \varepsilon^{1-s}\cdot C \to 0 \Rightarrow \int_{\varphi_4} g(z)\,dz \to 0$$

I^+:



$\varphi_3(t)=t\cdot(a+2\pi i\xi)$

$\varphi_3'(t)=a+2\pi i\xi$

$$\int_{\varphi_3} g(z)\,dz = -\int_\varepsilon^R (a+2\pi i\xi)^{1-s}\cdot\frac{1}{t^s}e^{-at}e^{-2\pi i\xi t}\,dt$$

$$\to -(a+2\pi i\xi)^{1-s}\, I^+$$

$\int_{\varphi_2} g(z)\,dz \to 0$ stejně.

$\int_{\varphi_1} g(z)\,dz \to \Gamma(1-s)$

$\int_{\varphi_4} g(z)\,dz \to 0$ stejně

Celkem: $\mathcal{F}(f)(\xi) = \Gamma(1-s)\cdot\left[(a-2\pi i\xi)^{s-1}+(a+2\pi i\xi)^{s-1}\right]$, $\xi\in\mathbb{R}$ ($\mathcal{F}(f)(\xi)$ je očividně sudá)

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10) $f(x)=\frac{1}{(x-\lambda)^m}$, $\lambda\in\mathbb{C}$, $\operatorname{Im}\lambda\neq 0$, $m\in\mathbb{N}$

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty}\frac{1}{(x-\lambda)^m}e^{-2\pi i x\xi}\,dx.$$

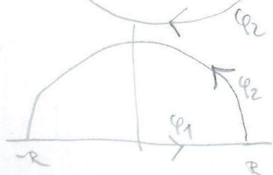
opačný směr probíhání!

Pro $\xi>0$:

, φ_2 Jordanem, $M_R\sim\frac{1}{R^m}$

$\alpha=-2\pi\xi<0 \to$ OK

Pro $\xi<0$:



φ_2 Jordanem, $\alpha=-2\pi\xi>0$

V obou případech je $\mathcal{F}(f)(\xi) = 2\pi i\,\operatorname{Res}_\lambda\left[\frac{e^{-2\pi i x\xi}}{(x-\lambda)^m}\right]$, pokud je λ uvnitř křivky

Možnosti: $\xi > 0,\ \operatorname{Im}\lambda > 0 \Rightarrow \mathcal{F}(f)(\xi) = 0$

$\xi < 0,\ \operatorname{Im}\lambda < 0 \Rightarrow \mathcal{F}(f)(\xi) = 0$

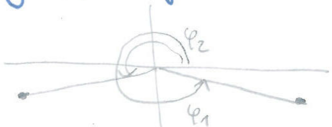
$\xi > 0,\ \operatorname{Im}\lambda < 0$ a $\xi < 0,\ \operatorname{Im}\lambda > 0 \Rightarrow \mathcal{F}(f)(\xi) = \frac{(-2\pi i \xi)^{m-1}}{(m-1)!} \cdot 2\pi i \cdot e^{-2\pi i \lambda \xi} \cdot \operatorname{sgn}(-\xi)$

$\xi = 0,\ m = 1$: integrál $\int_{-\infty}^{\infty} \frac{1}{x-\lambda}\,dx$ nekonverguje ($f \notin L^1$) ale ve smyslu $\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x-\lambda}\,dx =$

$= \lim_{R\to\infty} \ln(R-\lambda) - \ln(-R-\lambda)$. $\lambda = a+bi$: $\ln(R-a-bi) - \ln(-R-a-bi) = \underbrace{\ln\sqrt{\frac{R^2-2aR+a^2+b^2}{R^2+2aR+a^2+b^2}}}_{\to 0} + i\varphi_1 - i\varphi_2$

φ_1, φ_2 jsou úhly odpovídající $R-a-bi$ a $-R-a-bi$

Pro $b > 0$:



$R \to \infty$: $\varphi_1 \to 2\pi$, $\varphi_2 \to \pi$, $i\varphi_1 - i\varphi_2 \to i\pi$

Pro $b < 0$:

$R \to \infty$: $\varphi_1 \to 0$, $\varphi_2 \to \pi$, $i\varphi_1 - i\varphi_2 \to -i\pi$

Tj. $\xi = 0,\ m = 1$, $\mathcal{F}(f)(0) = i\pi \cdot \operatorname{sgn}(\operatorname{Im}\lambda)$

$m \geq 2;\ \xi = 0$: $\int_{-\infty}^{\infty} \frac{1}{(x-\lambda)^m}\,dx = 0$, protože $\lim_{x\to\pm\infty} \frac{1}{(x-\lambda)^{m-1}} = 0$

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11) $u_t - \Delta u = 0$ v $\mathbb{R}^+ \times \mathbb{R}^N$

$u(0,x) = u_0$ v $\mathbb{R}^N$

Ozn. $\hat{u}(t,\xi) = (\mathcal{F}_x u(t,\cdot))(\xi)$

$\hat{u}_0(\xi) = (\mathcal{F} u)(\xi)$

$$\Delta u = \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2} \Rightarrow \mathcal{F}(\Delta u)(\xi) = \sum_{j=1}^{N} (2\pi i)^2 \xi_j^2 \hat{u} = -4\pi^2 |\xi|^2 \hat{u}$$

$\Rightarrow$ Nový problém je obyč. dif. rce $\left.\begin{aligned} \hat{u}_t &= -4\pi^2|\xi|^2 \hat{u} \\ \hat{u}_0 &= \hat{u}(0) \end{aligned}\right\}$ separ. prom. $\Rightarrow \underline{\hat{u}(t,\xi) = \hat{u}_0 e^{-4\pi^2|\xi|^2 t}}$

$\Rightarrow u(t,x) = \int_{\mathbb{R}^N} \hat{u}_0(\xi) e^{-4\pi^2|\xi|^2 t} e^{2\pi i x\cdot\xi}\,d\xi$. To ale není nejvhodnější na práci. Lépe si všimneme, že $\hat{u}(t,\xi) = \hat{u}_0 \cdot \hat{G}$, kde $\hat{G} = e^{-4\pi^2|\xi|^2 t}$. Najdeme G a pak podle vzorce je $u(t,x) = u_0 * G$. Z příkladu 3, víme, že $\mathcal{F}(e^{-a|x|^2}) = \left(\frac{\pi}{a}\right)^{N/2} e^{-\frac{\pi^2|\xi|^2}{a}}$.

Volbou $a = \frac{1}{4t}$ dostáváme $e^{-\pi^2|\xi|^2 4t} = (4t\pi)^{-N/2} \mathcal{F}\left(e^{-\frac{|x|^2}{4t}}\right) = \mathcal{F}\left((4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}\right)$

$\Rightarrow G(t,x) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}$ a $\underline{\underline{u(t,x) = \int_{\mathbb{R}^N} \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\,dy}}$

12) $-y'' + a^2 y = f$ v $\mathbb{R}$. Ozn. $\hat{y}(\xi) = \mathcal{F}(y)(\xi)$, BÚNO $a > 0$.

$$-(2\pi i \xi)^2 \hat{y} + a^2 \hat{y} = \hat{f}$$

$$(4\pi^2\xi^2 + a^2)\hat{y} = \hat{f} \quad \Rightarrow \quad \hat{y} = \frac{\hat{f}}{4\pi^2\xi^2 + a^2} = \hat{f} \cdot \hat{G}, \text{ kde } \hat{G} = \frac{1}{4\pi^2\xi^2 + a^2}.$$

Z příkladu 6) víme, že $G(x) = \dfrac{e^{-a|x|}}{2a}$ $\Rightarrow$ $\underline{\underline{y(x) = \int_{-\infty}^{\infty} \frac{1}{2a} e^{-a|x-y|} f(y)\, dy}}$

To ale není vše, můžeme přičíst lib. řešení homogenní rovnice $y'' - a^2 y = 0$

$$y = C_1 e^{ax} + C_2 e^{-ax}$$

$$\Rightarrow \underline{\underline{y(x) = C_1 e^{ax} + C_2 e^{-ax} + \int_{-\infty}^{\infty} \frac{1}{2a} e^{-a|x-y|} f(y)\, dy}}$$