Sada příkladů 7/11

Funkce komplexní proměnné

Reziduová věta II

Použitím reziduové věty a pravidel pro počítání s rezidui spočtěte následující integrály: (ve smyslu Lebesgueově, Newtonově nebo ve smyslu hlavní hodnoty)

1. $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}t}{a+\cos t}, \quad a > 1$
2. $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}t}{(a+b\cos t)^2}, \quad a > b > 0$
3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 t\,\mathrm{d}t}{13+12\cos t}$
4. $\displaystyle\int_0^{\pi} \frac{\cos^4 t\,\mathrm{d}t}{1+\sin^2 t}$
5. $\displaystyle\int_0^{2\pi} \mathrm{e}^{\cos t}\cos\left((nt)-\sin t\right)\mathrm{d}t, \quad n \in \mathbb{N}$
6. $\displaystyle\int_{-\pi}^{\pi} \frac{\sin(nt)\ \mathrm{d}t}{1-2a\sin t+a^2}, \quad -1 < a < 1,\ n \in \mathbb{N}$
7. $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{(x+1)\sqrt{x}}$
8. $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{(x-1)\sqrt{x}}$
9. $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{x^a(x+b)}, \quad 0 < a < 1,\ b \neq 0$
10. $\displaystyle\int_0^{\infty} \frac{x^{a-1}\,\mathrm{d}x}{(x+1)(x+2)(x+3)}, \quad 0 < a < 3$
11. $\displaystyle\int_0^{1} \frac{x^{1-p}(1-x)^p\,\mathrm{d}x}{(1+x)^2}, \quad -1 < p < 2$

12. $\displaystyle\int_{-1}^{1} \frac{(1+x)^{1-p}(1-x)^p \, \mathrm{d}x}{x^2+1}, \quad -1 < p < 2$

13. $\displaystyle\int_{0}^{1} \frac{x^{1-p}(1-x)^p \, \mathrm{d}x}{x^2+1}, \quad -1 < p < 2$

14. $\displaystyle\int_{-1}^{1} \ln\left(\frac{1+x}{1-x}\right) \frac{\mathrm{d}x}{\sqrt[3]{(1-x)^2(1+x)}}$

15. $\displaystyle\int_{0}^{\infty} \frac{\ln x \, \mathrm{d}x}{x^2+a^2}$

16. $\displaystyle\int_{0}^{\infty} \frac{\ln x \, \mathrm{d}x}{(x-1)\sqrt{x}}$

17. $\displaystyle\int_{0}^{\infty} \frac{\ln x \, \mathrm{d}x}{(x+1)^2\sqrt[3]{x}}$

Sada příkladů 8/11

Funkce komplexní proměnné

Reziduová věta III

Použitím reziduové věty a pravidel pro počítání s rezidui spočtěte následující integrály: (ve smyslu Lebesgueově, Newtonově nebo ve smyslu hlavní hodnoty)

1. $\displaystyle\int_0^\infty \frac{\sin^2 x \,\mathrm{d}x}{x^2}$
2. $\displaystyle\int_0^\infty \frac{\sin^3 x \,\mathrm{d}x}{x^3}$
3. $\displaystyle\int_{-\infty}^\infty \frac{\mathrm{e}^{ax} \,\mathrm{d}x}{(\mathrm{e}^x + 1)(\mathrm{e}^x + 2)}, \qquad 0 < a < 2$
4. $\displaystyle\int_0^\infty \frac{\sin(ax)}{\sinh x} \,\mathrm{d}x, \qquad a \in \mathbb{R}$
5. $\displaystyle\int_0^\infty \frac{\cosh(ax)}{\cosh(\pi x)} \,\mathrm{d}x, \qquad |a| < \pi$
6. $\displaystyle\int_0^\infty \frac{\sin^2(ax)}{\cosh x} \,\mathrm{d}x, \qquad a > 0$

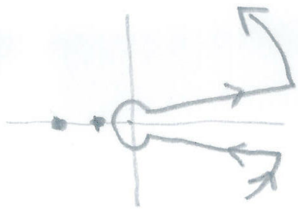
   Výsledky následujících příkladů vyjádřete pomocí funkce Γ

7. $\displaystyle\int_0^\infty x^z \mathrm{e}^{-x^2} \,\mathrm{d}x, \qquad \operatorname{Re} z > -1$
8. $\displaystyle\int_0^1 \left(\ln \frac{1}{t}\right)^{z-1} \mathrm{d}t, \qquad \operatorname{Re} z > 0$
9. $\displaystyle\int_0^\infty t^{z-1} \cos t \,\mathrm{d}t, \qquad 0 < \operatorname{Re} z < 1$
10. $\displaystyle\int_0^\infty \cos x^p \,\mathrm{d}x, \qquad p > 1$
11. $\displaystyle\int_0^\infty \frac{\sin x^p}{x^p} \,\mathrm{d}x, \qquad p > \frac{1}{2}$

11) $I = \int_0^1 \frac{x^{1-p}(1-x)^p}{(1+x)^2}dx$, $-1<p<2$

Uvědomme si: rozsah parametru p je dán požadavkem na konvergenci integrálu: $1-p > -1 \Rightarrow p<2$ a $p>-1$

Substitucí převedeme na známé případy: $\frac{x}{1-x} =: y \Rightarrow x = \frac{y}{1+y}$, $1+x = \frac{1+2y}{1+y}$



$dy = \left(\frac{1}{1-x} + \frac{x}{(1-x)^2}\right)dx = \frac{1}{(1-x)^2}dx$, $1-x = \frac{1}{1+y}$

$x\to 0: y\to 0$

$x\to 1: y\to\infty$, zobrazení je navíc prosté

$\Rightarrow I = \int_0^\infty y^{1-p}\cdot\frac{1}{1+y}\cdot\frac{(1+y)^2}{(1+2y)^2}\cdot\frac{1}{(1+y)^2}dy = \int_0^\infty \frac{y^{1-p}}{(1+y)(1+2y)^2}dy$. To už známe. Póly: $y=-1$ 1x, $y=-1/2$ 2x

Parametrizace cesty opět $\varphi_1: z = te^{i\eta}$ $t\in[\varepsilon, R]$ $\varphi_3: z = t\cdot e^{i(2\pi-\eta)}$ $t\in[R,\varepsilon]$

$\varphi_2: z = Re^{it}$ $t\in[\eta, 2\pi-\eta]$ $\varphi_4: z=\varepsilon e^{it}$ $t\in[2\pi-\eta,\eta]$

φ_2: $M_R \sim \frac{R^{1-p}}{R^3} = R^{-2-p}$, tj. $RM_R \sim R^{-1-p}$ a potřebujeme $-1-p<0 \Rightarrow \underline{p>-1}$

Jordanovo lemma: $\int_{\varphi_2} \to 0$

φ_4: dosadíme $z=\varepsilon e^{it}$, použijeme definici integrálu a v mocninách ε dostaneme: $\varepsilon\cdot\frac{\varepsilon^{1-p}}{1}$.

f je omezená na okolí počátku $\Rightarrow$ potřebujeme $\varepsilon^{2-p}\to 0 \Rightarrow 2-p>0 \Rightarrow \underline{p<2}$: $\int_{\varphi_4}\to 0$

$\int_{\varphi_1}\to I$, $\int_{\varphi_3}\frac{z^{1-p}}{(1+z)(1+2z)^2}dz \xrightarrow{\eta\to 0+} \int_R^\varepsilon \frac{t^{1-p}\cdot e^{2\pi i(1-p)}}{(1+t)(1+2t)^2}dt \xrightarrow{R,\varepsilon} -e^{2\pi i(1-p)}I = -e^{-2\pi i p}I$

$\Rightarrow I(1-e^{2\pi i(1-p)}) = 2\pi i(\mathrm{Res}_{-1} f + \mathrm{Res}_{-1/2} f)$

$= 2\pi i \cdot e^{-\pi i p}\cdot(-1 - p2^{p-1} + 2^p)$

$\mathrm{Res}_{-1} f = \frac{(-1)^{1-p}}{(-1)^2} = e^{\pi i(-1-p)} = -e^{-\pi i p}$

$\mathrm{Res}_{-1/2} f = \left(\frac{z^{1-p}}{4(1+z)}\right)'\Big|_{z=-1/2} = \left[(1-p)\frac{z^{-p}}{4(1+z)} - \frac{z^{1-p}}{4(1+z)^2}\right]_{z=-\frac12}$

$= \frac{1-p}{2}\cdot\left(-\frac12\right)^{-p} - \left(-\frac12\right)^{(1-p)} = e^{-\pi i p}\cdot\left[\frac{1-p}{2^{-p+1}} + \frac{1}{2^{1-p}}\right]$

$\Rightarrow I = (-1 - p2^{p-1} + 2^p)\cdot\frac{2\pi i}{e^{\pi i p} - e^{-\pi i p}} = \underline{\underline{\frac{\pi(2^p - p2^{p-1} - 1)}{\sin\pi p}}}$

p=0 a p=1 třeba dopočítat zvlášť!

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12) $I = \int_{-1}^1 \frac{(1+x)^{1-p}(1-x)^p}{x^2+1}dx$, $-1<p<2$. Stejně jako 11) $y = \frac{1+x}{1-x} \Rightarrow x = \frac{y-1}{y+1}$ $x\to -1 \Rightarrow y\to 0$, $x\to 1\Rightarrow y\to\infty$

$dy = \frac{2}{(1-x)^2}dx$, $1-x = \frac{2}{y+1}$

$I = \int_0^\infty y^{1-p}\cdot\frac{2}{y+1}\cdot\frac{1}{\frac{(y-1)^2+(y+1)^2}{(y+1)^2}}\cdot\frac{2}{(y+1)^2}dy = \int_0^\infty y^{1-p}\cdot\frac{2}{(y+1)(y^2+1)}dy$. Póly: $y=-1$ 1x, $y=\pm i$ 1x

Cesta totožná, $\int_{\varphi_2}\to 0$, $\int_{\varphi_4}\to 0$.

$I(1-e^{-2\pi i p}) = 2\pi i(\mathrm{Res}_{-1} f + \mathrm{Res}_{i} f + \mathrm{Res}_{-i} f) = 2\pi i\left(\frac{-2\cdot e^{-\pi i p}}{2} + \frac{2i\cdot e^{-\frac{\pi i p}{2}}}{(1+i)\cdot 2i} + \frac{-2i\cdot e^{-\frac32\pi i p}}{(1-i)\cdot(-2i)}\right) = 2\pi i e^{-\pi i p}(\dots)$

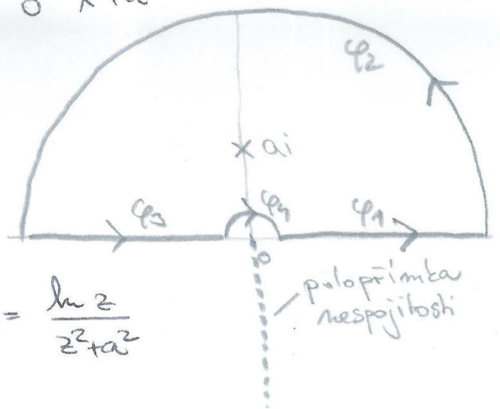
Platí: $\frac{e^{\frac{\pi i p}{2}}}{1+i} + \frac{e^{-\frac{\pi i p}{2}}}{1-i} = \frac{(e^{\frac{\pi i p}{2}} + e^{-\frac{\pi i p}{2}}) - i(e^{\frac{\pi i p}{2}} - e^{-\frac{\pi i p}{2}})}{2} = \cos\frac{\pi p}{2} + \sin\frac{\pi p}{2}$

$\Rightarrow I = \frac{\pi}{\sin(\pi p)}\cdot\left[-1+\cos\frac{\pi p}{2}+\sin\frac{\pi p}{2}\right]$

Opět p=0 a p=1 třeba dopočítat zvlášť!

14, Uděláme níže, nejdříve 15, -17,

15, $\displaystyle\int_0^\infty \frac{\ln x\,dx}{x^2+a^2} = I$. Potřebujeme, ať fce, která násobí $\ln x$ je sudá. Pak volíme toto



Volíme komplexní logaritmus $\ln z = \ln|z| + i\arg z$,

kde $\arg z \in \left(-\frac{\pi}{2}, \frac{3}{2}\pi\right)$

$\varphi_1 : z = t,\ t\in[\varepsilon, R]$ $\qquad \varphi_3 : z = t,\ t\in[-R,-\varepsilon]$

$\varphi_2 : z = Re^{it},\ t\in[0,\pi]$ $\qquad \varphi_4 : z = \varepsilon e^{it},\ t\in[\pi, 0]$

$f(z) = \dfrac{\ln z}{z^2+a^2}$

φ_2: $M_R \sim \dfrac{1}{R^2}\ln R$, $RM_R \sim \dfrac{\ln R}{R}$ a dle Jordana

$\displaystyle\int_{\varphi_2} \to 0$

φ_1: $\displaystyle\int_{\varphi_1} \to I$

φ_4: Opět z definice integrálu dostáváme $\varepsilon\cdot\ln\varepsilon\cdot \underbrace{\max \frac{1}{z^2+a^2}}_{\leq C}$

a protože $\varepsilon\ln\varepsilon \to 0$, $\displaystyle\int_{\varphi_4} \to 0$

$$\varphi_3: \int_{\varphi_3} \frac{\ln z}{z^2+a^2} = \int_{-R}^{-\varepsilon} \frac{\ln t}{t^2+a^2}dt = \int_{-R}^{-\varepsilon}\frac{\ln|t| + i\pi}{t^2+a^2}dt = \underbrace{\int_\varepsilon^R \frac{\ln t\,dt}{t^2+a^2}}_{\to I} + i\pi\int_\varepsilon^R \frac{dt}{t^2+a^2}$$

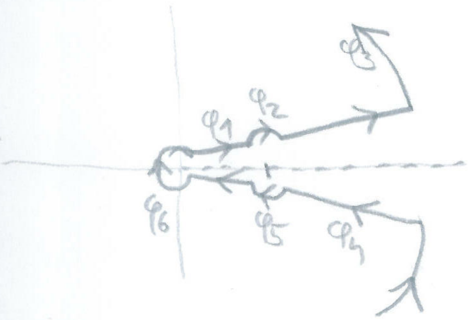
sudost

$$\text{Celkem: } 2I + \frac{i\pi}{2}\underbrace{\int_{-\infty}^{\infty}\frac{dt}{t^2+a^2}}_{\hookrightarrow \text{ Umíme !!}} = 2\pi i \operatorname{Res}_{ai} f$$

$$2I + \frac{i\pi}{2}\cdot 2\pi i \operatorname{Res}_{ai}\frac{1}{z^2+a^2} = 2\pi i\operatorname{Res}_{ai}\frac{\ln z}{z^2+a^2}$$

$$I = \pi i\frac{\ln ai}{2ai} + \frac{\pi^2}{2}\cdot\frac{1}{2ai} = \frac{\pi}{2a}\cdot\left(\ln a + \frac{i\pi}{2} - \frac{i\pi}{2}\right) = \underline{\underline{\frac{\pi\ln a}{2a}}}$$

16, $I = \displaystyle\int_0^\infty \frac{\ln x\,dx}{(x-1)\sqrt{x}}$. Fce bez $\ln x$ není sudá $\Rightarrow$ postup z 15, nefunguje. Pomůže nám odmocnina. Polopřímka nespojitosti $\ln z$ bude zde kladná reálná poloosa.



$\displaystyle\int_{\varphi_1} f(z)dz \to I$

$\displaystyle\int_{\varphi_2} f$: Obkružujeme shora : LOOPN1 : $\to iAb$, kde $b\to\pi$

$A = \displaystyle\lim_{\text{shora}} \frac{\ln z}{\sqrt{z}} = \ln 1 = 0 \Rightarrow \int_{\varphi_2} \to 0$

$\displaystyle\int_{\varphi_3}$: Jordan: $M_R \sim \dfrac{\ln R}{R^{3/2}}$, $RM_R \sim \dfrac{\ln R}{R^{1/2}} \to 0 \Rightarrow \displaystyle\int_{\varphi_3}\to 0$

$\displaystyle\int_{\varphi_5} f$: LOOPN1 : $\to iAb$, kde $b\to-\pi$ a $A = \displaystyle\lim_{\text{zdola}}\frac{\ln z}{\sqrt{z}} = \frac{\ln 1 + 2\pi i}{-1} = -2\pi i$

$\Rightarrow \displaystyle\int_{\varphi_5} f \to i(-2\pi i)\cdot(-\pi) = -2\pi^2$

$\displaystyle\int_{\varphi_6} f$: Definice křivkového int.: $\left|\displaystyle\int_{\varphi_6} f\right| \leq \dfrac{\varepsilon\ln\varepsilon}{\sqrt{\varepsilon}}\cdot\underbrace{\max\left|\frac{1}{z-1}\right|}_{\leq C} \xrightarrow{\varepsilon\to 0} 0$

$\int_{\varphi_4} f$: $\ln z \to \ln t + 2\pi i$, $\sqrt{z} \to \sqrt{t}\cdot e^{\frac{2\pi i}{2}} = -\sqrt{t}$ $\Bigg\}$ $\int_{\varphi_4} f \to -\int_0^\infty \frac{\ln t + 2\pi i}{(t-1)\cdot(-\sqrt{t})}\,dt = I + 2\pi i \int_0^\infty \frac{dt}{\sqrt{t}(t-1)}$

opačný směr

Celkem: $2I + 2\pi i \underbrace{\int_0^\infty \frac{dt}{\sqrt{t}(t-1)}}_{J} - 2\pi^2 = 0$ (f je holomorfní uvnitř Γ)

ale pozor! ve smyslu hlavní hodnoty!!

Zbývající integrál buď opět reziduovou větou nebo přímo: $y = \sqrt{t}$, $dy = \frac{1}{2\sqrt{t}}\,dt$

$\Rightarrow J = \text{p.v.}\int_0^\infty \frac{2\,dy}{y^2-1} = \text{p.v.}\int_0^\infty \frac{1}{y-1} - \frac{1}{y+1}\,dy$

$= \lim_{\varepsilon\to 0}\left[\ln\frac{1-y}{1+y}\right]_0^{1-\varepsilon} + \left[\ln\frac{y-1}{y+1}\right]_{1+\varepsilon}^{\infty} = \lim_{\varepsilon\to 0_+} \frac{\ln\varepsilon}{2-\varepsilon} - \frac{\ln\varepsilon}{2+\varepsilon}$

$= \lim_{\varepsilon\to 0_+} \frac{2\varepsilon\ln\varepsilon}{4-\varepsilon^2} = 0 \Rightarrow J = 0$ a $\underline{I = \pi^2}$

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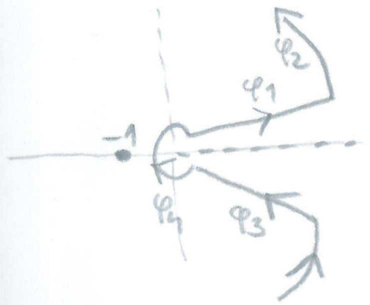
17) $I = \int_0^\infty \frac{\ln x\,dx}{(x+1)^2\sqrt[3]{x}}$ $\qquad \int_{\varphi_1} \to I$ $\qquad f(z) = \frac{\ln z}{(z+1)^2 z^{1/3}}$. Nespojitosti na $\mathbb{R}^+$



$\int_{\varphi_2}$: Jordan $M_R \sim \frac{\ln R}{R^{2+1/3}}$, $RM_R \sim \frac{\ln R}{R^{4/3}} \to 0 \Rightarrow \int_{\varphi_2} \to 0$

$\int_{\varphi_4}$: Máme $\varepsilon\cdot\ln\varepsilon\cdot\varepsilon^{-1/3}\cdot \underbrace{\max_{|z|\le\varepsilon}\left|\frac{1}{(z+1)^2}\right|}_{\le c} \xrightarrow{\varepsilon\to 0_+} 0 \Rightarrow \int_{\varphi_4} \to 0$

$\int_{\varphi_3}$: $\ln z \to \ln t + 2\pi i$, $\sqrt[3]{z} \to \sqrt[3]{t}\cdot e^{\frac{2\pi i}{3}}$ $\Bigg\}$ $\int_{\varphi_3} f \to -\int_0^\infty \frac{\ln t + 2\pi i}{(t+1)^2\sqrt[3]{t}}\cdot e^{-\frac{2\pi i}{3}}\,dt$

$= -e^{-\frac{2\pi i}{3}} I - 2\pi i\, e^{-\frac{2\pi i}{3}} \int_0^\infty \frac{1}{(t+1)^2\sqrt[3]{t}}\,dt$

$\Rightarrow I\cdot\left(1 - e^{-\frac{2\pi i}{3}}\right) - 2\pi i\, e^{-\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{dt}{(t+1)^2\sqrt[3]{t}}}_{J} = 2\pi i \operatorname{Res}_{-1} f = 2\pi i \left(\frac{\ln z}{\sqrt[3]{z}}\right)'\Bigg|_{z=-1}$

$= 2\pi i\left[\frac{1}{z\sqrt[3]{z}} - \frac{1}{3}\frac{\ln z}{z^{4/3}}\right]\Bigg|_{z=-1} = 2\pi i(-1)\cdot e^{-\frac{\pi i}{3}}\cdot\left(1 - \frac{i\pi}{3}\right)$

$= -2\pi i\, e^{-\frac{\pi i}{3}}\cdot\left(1 - \frac{i\pi}{3}\right)$

J ... opět reziduovou větou, stejná cesta

$J\cdot\left(1 - e^{-\frac{2\pi i}{3}}\right) = 2\pi i\cdot\left(z^{-1/3}\right)'\Big|_{z=-1} = 2\pi i\cdot\left(-\frac{1}{3}\right)\cdot(-1)^{-4/3} = \frac{2\pi i}{3}\cdot e^{-\frac{\pi i}{3}}$

$\Rightarrow J = \frac{\pi}{3\cdot\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$

$\Rightarrow I\cdot\left(\sin\frac{\pi}{3}\right) - \pi\cdot e^{-\frac{\pi i}{3}}\cdot\frac{2\pi}{3\sqrt{3}} = -\pi\left(1 - \frac{i\pi}{3}\right)$

$I\cdot\frac{\sqrt{3}}{2} - \frac{2\pi^2}{3\sqrt{3}}\cdot\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -\pi + \frac{i\pi^2}{3} \Rightarrow \underline{I = -\frac{2\pi}{\sqrt{3}} + \frac{2\pi^2}{9}}$

4

14) $I = \int_{-1}^{1} \ln\left(\frac{1+x}{1-x}\right) \frac{dx}{\sqrt[3]{(1-x)^2(1+x)}}$

Substituce $y = \frac{1+x}{1-x}$ $x \to -1 \Rightarrow y \to 0$, $x \to +1 \Rightarrow y \to +\infty$

$dy = \left(\frac{1}{1-x} + \frac{1+x}{(1-x)^2}\right) dx = \frac{2}{(1-x)^2} dx$

$x = \frac{y-1}{y+1}$ $1-x = \frac{2}{y+1}$

$$I = \int_0^\infty \ln y \cdot y^{-1/3} \cdot \left(\frac{y+1}{2}\right) \cdot \frac{4}{(y+1)^2} \cdot \frac{1}{2}\, dy$$

$$= \int_0^\infty \frac{\ln y \, dy}{y^{1/3} \cdot (y+1)}$$

To je totéž jako příklad 17), jen v $y=-1$ je jednonásobný pól místo dvojnásobného. Stejná cesta, stejný postup. Dostaneme

$$I \cdot \left(1 - e^{-\frac{2\pi i}{3}}\right) - 2\pi i\, e^{-\frac{2\pi i}{3}} \underbrace{\int_0^\infty \frac{dt}{t^{1/3}(t+1)}}_{J} = 2\pi i \operatorname{Res}_{-1} f = 2\pi i \frac{\ln(-1)}{(-1)^{1/3}} = 2\pi i \left(i\pi \cdot e^{-i\pi/3} \right) = -2\pi^2 \cdot e^{-i\pi/3}$$

$J = \int_0^\infty \frac{dt}{t^{1/3}(t+1)}$ buď spočítáme stejnou integrační cestou nebo rovnou použijeme výsledek příkladu 9) s $a = 1/3$, $b = 1$ $\Rightarrow J = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$

$$\Rightarrow I \cdot 2i \sin\frac{\pi}{3} - 2\pi i \cdot \frac{2\pi}{\sqrt{3}} \cdot \left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right) = -2\pi^2$$

$$I \cdot 2i \cdot \frac{\sqrt{3}}{2} = i \frac{2\pi^2}{\sqrt{3}} \Rightarrow \underline{\underline{I = \frac{2}{3}\pi^2}}$$

Poslední sada příkladů: různé méně standardní příklady, kde se naučíme další triky

1) $I = \int_0^\infty \underbrace{\frac{\sin^2 x}{x^2}}_{\text{sudá}} dx$ $\sin x = \frac{e^{ix} - e^{-ix}}{2i} \Rightarrow \sin^2 x = \frac{e^{2ix} - 2 + e^{-2ix}}{-4}$ ... Nelze počítat najednou kvůli Jordanovi, viz níže

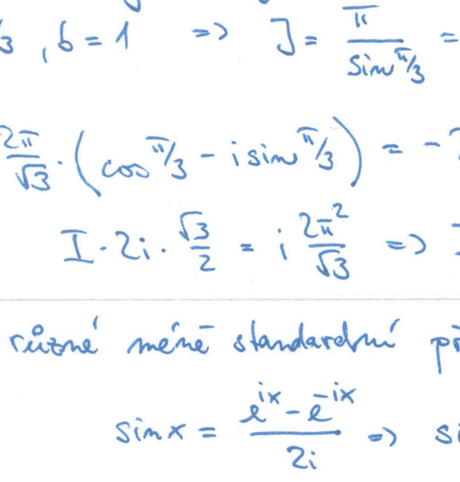
$$\Rightarrow I = -\frac{1}{8}\left(\underbrace{\int_{-\infty}^{\infty} \frac{e^{2ix} - 1}{x^2} dx}_{I_1} + \underbrace{\int_{-\infty}^{\infty} \frac{e^{-2ix} - 1}{x^2} dx}_{I_2} \right)$$

Původní fce $\frac{\sin^2 x}{x^2}$ má v $x=0$ konečnou limitu

Nové fce $\frac{e^{\pm 2ix} - 1}{x^2}$ mají v $x=0$ pól, ale naštěstí jen 1-násobný, protože

$$\lim_{x \to 0} x \cdot \frac{e^{\pm 2ix} - 1}{x^2} = \pm 2i \neq 0$$

I_1: $f(z) = \frac{e^{2iz} - 1}{z^2}$



$$\lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \int_{\varphi_1} f(z)\,dz = \text{p.v.} \int_{-\infty}^{\infty} f(z)\,dz = I_1$$

φ_2: Jordanovo lemma
- pro fci $-\frac{1}{z^2}$ je $\alpha = 0$, $M_R \sim \frac{1}{R^2}$, $R M_R \to 0$ OK
- pro fci $\frac{e^{2iz}}{z^2}$ je $\alpha = 2$, $M_R \sim \frac{1}{R^2} \to 0$ OK

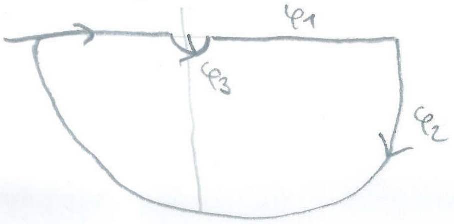
$\int_{\varphi_2} f \xrightarrow{R \to +\infty} 0$

φ_3: $\int_{\varphi_3} f(z)\,dz \xrightarrow{\varepsilon \to 0} iAb$ dle LOOPN1, kde $b = -\pi$ a $A = \lim_{z \to 0 \,(\text{shora})} z \cdot \frac{e^{2iz} - 1}{z^2} = 2i$

$$\Rightarrow \int_{\varphi_3} f(z)\,dz \to i(2i) \cdot (-\pi) = 2\pi \quad \Rightarrow I_1 + 0 + 2\pi = 0 \Rightarrow I_1 = -2\pi$$

Cauchyova věta, f je holomorfní

I_2 analogicky, jen pro Jordana potřebujeme $\operatorname{Im} z < 0$, protože máme e^{-2iz}



Opět $\int_{\varphi_1} f(z)\,dz \to I_2$

$\int_{\varphi_2} f$ dle Jordana $\left\langle \begin{array}{l} \alpha = 0,\ M_R \sim \frac{1}{R^2} \text{ pro } -\frac{1}{z^2} \\ \alpha = -2,\ M_R \sim \frac{1}{R^2} \text{ pro } \frac{e^{-2iz}}{z^2} \end{array} \right\} \to 0$

$\int_{\varphi_3} f \to iAb$ dle LOOPN1, kde $b = \pi$, $A = \lim_{z\to 0} z \cdot \frac{e^{-2iz}-1}{z^2} =$
$= -2i$

$\Rightarrow \int_{\varphi_3} f \to i\cdot(-2i)\cdot\pi = 2\pi$

$I_2 + 0 + 2\pi = 0$ ← Cauchy

$\Rightarrow I_2 = -2\pi$

$\Rightarrow I = -\frac{1}{8}(-2\pi - 2\pi) = \underline{\underline{\frac{\pi}{2}}}$

2) $\int_0^\infty \frac{\sin^3 x}{x^3} = I$

$\sin^3 x = -\frac{1}{8i}\left(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}\right)$

$= -\frac{1}{8i}\left(e^{ix}\cdot(e^{2ix}-1) - 2e^{ix} - e^{-ix}(e^{-2ix}-1) + 2e^{-ix}\right)$

$\frac{e^{2ix}-1}{x^3}$ má pořád pól nás. 2
$\Rightarrow$ musíme upravovat dál:

$= -\frac{1}{8i}\Big(e^{ix}(e^{2ix}-1-2ix) + 2e^{ix}(e^{-ix}+ix-1)$
$- e^{-ix}(e^{-2ix}-1+2ix) - 2e^{-ix}(e^{ix}-1-ix)\Big)$

Teď jsem vytvořil póly násobnosti 1.

Opět ze sudosti $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\,dx = -\frac{1}{16i}\left(I_1 + I_2 + I_3 + I_4\right)$

$I_1 = \int_{-\infty}^{\infty} e^{ix}\cdot\frac{e^{2ix}-1-2ix}{x^3}\,dx$ .... integrujeme přes horní půlkruh s odcházením pólu v nule

$\int_{\varphi_1} f \to I_1$, $\int_{\varphi_2}$ Jordan: $\frac{e^{3iz}}{z^3}$ OK $\alpha = 3$, $M_R \sim \frac{1}{R^3}$, $-\frac{e^{iz}}{z^3}$ OK, $\alpha = 1$, $M_R \sim \frac{1}{R^3}$

$\frac{-2ie^{iz}}{z^2}$ OK, $\alpha = 1$, $M_R \sim \frac{1}{R^2}$ $\quad \int_{\varphi_2} f \to 0$

$\int_{\varphi_3} f \to iAb$ dle LOOPN1: $b = -\pi$, $A = \lim_{z\to 0} \underbrace{e^{iz}}_{\downarrow\, 1} \cdot \underbrace{\frac{e^{2iz}-1-2iz}{z^3}}_{\text{Taylor}} \cdot z = \frac{(2i)^2}{2!} = -2$

$\Rightarrow I_1 + 0 + 2\pi i = 0$ dle Cauchyho (f holomorfní) $\Rightarrow I_1 = -2\pi i$

$I_2 = \int_{-\infty}^{\infty} \frac{2e^{ix}(e^{-ix}+ix-1)}{x^3}\,dx$ ... Stejná cesta: $\int_{\varphi_1} f \to I_2$

$\int_{\varphi_2} f$ Jordan: $\frac{2}{z^3}$ OK, $\alpha = 0$, $M_R \sim \frac{1}{R^3}$. $\frac{2ie^{iz}}{z^2}$ OK, $\alpha = 1$, $M_R \sim \frac{1}{R^2}$ $\quad -\frac{2e^{iz}}{z^3}$ OK, $\alpha = 1$, $M_R \sim \frac{1}{R^3}$

$\Rightarrow \int_{\varphi_2} f \to 0$

$\int_{\varphi_3} f \to iAb$ dle LOOPN1: $b = -\pi$, $A = \lim_{z\to 0} 2\cdot e^{iz}\cdot\frac{e^{-iz}-1+iz}{z^3}\cdot z = 2\cdot 1\cdot\frac{(-i)^2}{2!} = -1$

$\Rightarrow I_2 + 0 + \pi i = 0 \Rightarrow I_2 = -\pi i$

I_3 a I_4 analogicky, jen musíme kvůli Jordanovi integrovat přes dolní polorovinu

Podobně jako v předchozím příkladě.

$$I_3: \int_{\varphi_1} \to I_3, \quad \int_{\varphi_2} \text{Jordan} \to 0, \quad \int_{\varphi_3} f \to iAb, \quad b=\pi, \quad A = \lim_{z\to 0} -e^{-iz}\cdot\left(\frac{e^{-2iz}-1+2iz}{z^3}\right)\cdot z = -\frac{(-2i)^2}{2!} = 2$$

$$\Rightarrow I_3 + 0 + 2\pi i = 0 \Rightarrow I_3 = -2\pi i$$

$$I_4: \int_{\varphi_1} \to I_4, \quad \int_{\varphi_2} \text{Jordan} \to 0, \quad \int_{\varphi_3} f \to iAb, \quad b=\pi, \quad A = \lim_{z\to 0} -2\cdot e^{-iz}\cdot\left(\frac{e^{iz}-1-iz}{z^3}\right)\cdot z = -2\cdot\frac{i^2}{2!} = 1$$

$$\Rightarrow I_4 + 0 + \pi i = 0 \Rightarrow I_4 = -\pi i$$

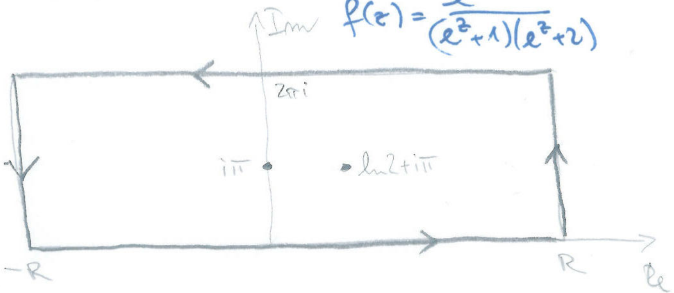
Celkem $I = -\frac{1}{16i}\left(-6\pi i\right) = \underline{\underline{\frac{3}{8}\pi}}$

3) $\int_{-\infty}^{\infty} \frac{e^{ax}}{(e^x+1)(e^x+2)} = I \quad 0<a<2$

Póly: $e^x = -1 = e^{i\pi + 2k\pi i} \Rightarrow x = i(\pi + 2k\pi)$

$e^x = -2 = e^{\ln 2 + i\pi + 2k\pi i} \Rightarrow x = \ln 2 + i(\pi + 2k\pi)$

$f(z) = \frac{e^{az}}{(e^z+1)(e^z+2)}$



$\varphi_1(t) = t, \quad t\in[-R,R]$

$\varphi_2(t) = R+it, \quad t\in[0,2\pi]$

$\varphi_3(t) = t+2\pi i, \quad t\in[R,-R]$

$\varphi_4(t) = -R+it, \quad t\in[2\pi,0]$

$\int_{\varphi_1} f(z)dz \to I$

$$\int_{\varphi_2} f(z)dz = \int_0^{2\pi} \frac{e^{aR}\cdot e^{ait}}{(e^R\cdot e^{it}+1)(e^R\cdot e^{it}+2)}\cdot i\,dt \qquad \left|\int_{\varphi_2} f\right| \le \int_0^{2\pi}\frac{e^{aR}}{(e^R-1)(e^R-2)}dt \le 2\pi\cdot e^{R\cdot(a-2)}\cdot C \xrightarrow[R\to\infty]{} 0 \text{ protože } a<2$$

$$\int_{\varphi_3} f(z)dz = -\int_{-R}^{R}\frac{e^{at}\cdot e^{2\pi i a}}{(e^t+1)(e^t+2)}dt = -e^{2\pi i a}\int_{\varphi_1} f(z)dz \to -e^{2\pi i a} I$$

$$\int_{\varphi_4} f(z)dz = -\int_0^{2\pi}\frac{e^{-aR}\cdot e^{ait}}{(e^{-R}\cdot e^{it}+1)(e^{-R}e^{it}+2)}\,i\,dt \Rightarrow \left|\int_{\varphi_4} f\right| \le \int_0^{2\pi}\frac{e^{-aR}}{\underbrace{(1-e^{-R})}_{\ge 1/2,\ \to 1}\underbrace{(2-e^{-R})}_{\ge 1,\ \to 2}}dt \le 4\pi e^{-aR} \to 0 \text{ protože } a>0$$

Celkem: $I\cdot\left(1-e^{2\pi i a}\right) = \left(\operatorname{Res}_{i\pi} f + \operatorname{Res}_{\ln 2+i\pi} f\right)\cdot 2\pi i$

Pozor na výpočet reziduí! Nemáme tvar $z-z_0$ ve jmenovateli!

$$\operatorname{Res}_{i\pi} f = \frac{e^{ai\pi}}{(e^{i\pi}+2)}\cdot\frac{1}{(e^z+1)'}\Big|_{z=i\pi} = \frac{e^{ai\pi}}{1}\cdot\frac{1}{e^{i\pi}} = -e^{ai\pi}$$

Podobně $\operatorname{Res}_{\ln 2+i\pi} f = \frac{e^{ai\pi}\cdot 2^a}{-1}\cdot\frac{1}{(e^z+2)'}\Big|_{z=\ln 2+i\pi} = -2^a e^{ai\pi}\cdot\left(-\frac{1}{2}\right) = 2^{a-1}e^{ai\pi}$

$$I\left(1-e^{2\pi i a}\right) = 2\pi i\, e^{i\pi a}\cdot\left(2^{a-1}-1\right)$$

$$\underline{\underline{I = \frac{\pi}{\sin(\pi a)}\cdot\left(1-2^{a-1}\right)}}$$

To ale nefunguje pro $a=1$. Tam musíme počítat metodami 1. semestru

$$\int_{-\infty}^{\infty} \frac{e^x\,dx}{(e^x+1)(e^x+2)} \qquad \left.\begin{array}{ll} t=e^x & x\to-\infty \Rightarrow t\to 0 \\ dt=e^x\,dx & x\to+\infty \Rightarrow t\to\infty \end{array}\right\} \Rightarrow I=\int_0^\infty \frac{dt}{(t+1)(t+2)}$$

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \end{array}$$

$$\Rightarrow I = \left[\ln\frac{t+1}{t+2}\right]_0^\infty = \ln 1 - \ln\frac{1}{2} = \underline{\underline{\ln 2}}$$

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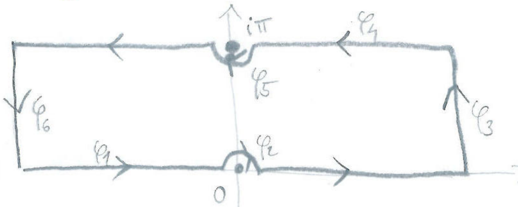
(4) $I=\int_0^\infty \frac{\sin(ax)}{\sinh x}\,dx$, $a\in\mathbb{R}$. Sudost: $I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(ax)}{\sinh x}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{e^{iax}}{\frac{e^x-e^{-x}}{2}}\,dx$

$$= \mathrm{Im}\underbrace{\int_{-\infty}^{\infty}\frac{e^{iax}}{e^x-e^{-x}}\,dx}_{=:J} \;\ldots\text{ zde už integrál jako p.v.}$$

Póly $f(z)=\frac{e^{iaz}}{e^z-e^{-z}}$: $e^z=e^{-z} \Leftrightarrow e^{2z}=1=e^{2k\pi i} \Rightarrow$ póly jsou $z_k=k\pi i$, $k\in\mathbb{Z}$



$\varphi_1: z=t,\ t\in[-R,-\varepsilon]\cup[\varepsilon,R]$ $\qquad$ $\varphi_4: z=t+i\pi,\ t\in[R,\varepsilon]\cup[-\varepsilon,-R]$

$\varphi_2: z=\varepsilon e^{it},\ t\in[\pi,0]$ $\qquad$ $\varphi_5: z=i\pi+\varepsilon e^{it},\ t\in[0,-\pi]$

$\varphi_3: z=R+it,\ t\in[0,\pi]$ $\qquad$ $\varphi_6: z=-R+it,\ t\in[\pi,0]$

$\int_{\varphi_1} f \to J$, $\int_{\varphi_2} f \to \underset{-\frac{i\pi}{2}}{\overset{}{iAb}}$ dle LOOPN1, $b=-\pi$, $A=\lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{z}{e^z-e^{-z}} = \lim \frac{1}{e^{-z}}\cdot\frac{2z}{e^{2z}-1}\cdot\frac{1}{2}=\frac{1}{2}$

$$\left|\int_{\varphi_3} f\right| \le \int_0^\pi \frac{e^{-at}}{|e^R\cdot(e^{it}-e^{-2R}e^{it})|}\,dt \le \frac{C}{e^R}\xrightarrow{R\to\infty}0, \quad \int_{\varphi_4} f \to -\int_{-\infty}^{\infty}\frac{e^{ia(t+i\pi)}}{e^t\cdot e^{i\pi}-e^{-t}\cdot e^{-i\pi}}\,dt = e^{-a\pi}\cdot J$$

$\int_{\varphi_2} f \to iAb$ dle LOOPN1: , $b=-\pi$, $A=\lim_{z\to i\pi} f(z)(z-i\pi) = e^{iai\pi}\lim_{z\to i\pi}\frac{z-i\pi}{e^z-e^{-z}} \overset{\text{l'Hosp}}{=} e^{-a\pi}\cdot\lim_{z\to i\pi}\frac{1}{e^z+e^{-z}} = -\frac{e^{-a\pi}}{2}$

$$iAb = \frac{e^{-a\pi}\,i\pi}{2}$$

$$\left|\int_{\varphi_6} f\right| \le \int_0^\pi \frac{e^{-at}}{e^R\cdot|e^{it}-e^{-2R}e^{it}|}\,dt \le \frac{C}{e^R}\to 0$$

DOHROMADY: $J\left(1+e^{-a\pi}\right)-\frac{i\pi}{2}\left(1-e^{-a\pi}\right)=0$

$$J=\frac{i\pi}{2}\left(\frac{1-e^{-a\pi}}{1+e^{-a\pi}}\right)$$

$$I=\mathrm{Im}\,J = \underline{\underline{\frac{\pi}{2}\cdot\tanh\frac{a\pi}{2}}}$$

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(5) $I=\int_0^\infty\frac{\cosh(ax)}{\cosh(\pi x)}\,dx$, $|a|<\pi$ $\quad$ Sudost: $I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ax}+e^{-ax}}{e^{\pi x}+e^{-\pi x}}\,dx$ $\quad$ Označme $J_a=\int_{-\infty}^{\infty}\frac{e^{ax}}{e^{\pi x}+e^{-\pi x}}\,dx$

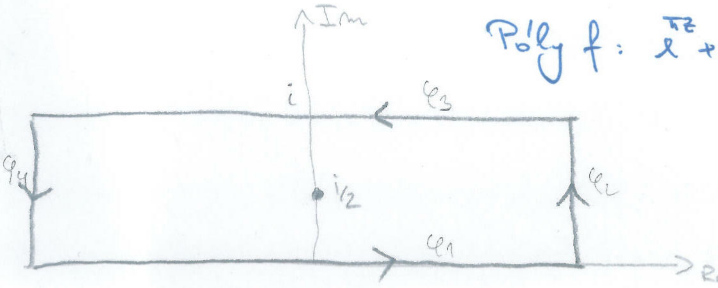
Lehce nahlédneme, že $J_a=J_{-a}$: V J_a uděláme substituci $y=-x$; $dy=-dx$, $J_a=\int_{\infty}^{-\infty}\frac{e^{-ay}}{e^{-\pi y}+e^{\pi y}}\cdot(-1)\,dy$

$$=\int_{-\infty}^{\infty}\frac{e^{-ay}}{e^{\pi y}+e^{-\pi y}}\,dy = J_{-a}$$

Protože $I=\frac{1}{2}(J_a+J_{-a})$, je též $I=J_a$. $f(z):=\frac{e^{az}}{e^{\pi z}+e^{-\pi z}}$

Póly f: $e^{\pi z}+e^{-\pi z}=0 \Rightarrow e^{2\pi z}=-1=e^{i\pi+2k\pi i}$

$\Rightarrow z_k=\frac{i}{2}+ki$



$\varphi_1: z=t,\ t\in[-R,R]$ $\qquad$ $\varphi_3: z=t+i,\ t\in[R,-R]$

$\varphi_2: z=R+it,\ t\in[0,1]$ $\qquad$ $\varphi_4: z=-R+it,\ t\in[1,0]$

$\int_{\varphi_1} f \to J_a$, $\int_{\varphi_3} f(z)\,dz = \int_R^{-R} \frac{e^{a(i+t)}}{e^{\pi(i+t)}+e^{-\pi(i+t)}}\,dt = e^{ai}\int_R^{-R}\frac{e^{at}}{-e^{t\pi}-e^{-\pi t}}\,dt \to e^{ai} J_a$

$\left|\int_{\varphi_2} f\right| \le \int_0^1 \frac{e^{aR}}{e^{\pi R}\cdot\left|e^{\pi i t}+e^{-2\pi R}e^{-\pi i t}\right|}\,dt \le C e^{(a-\pi)R} \to 0$ pro $a<\pi$

$\left|\int_{\varphi_4} f\right| \le \int_0^1 \frac{e^{-aR}}{e^{\pi R}\cdot\left|e^{-\pi i t}+e^{-2\pi R}e^{\pi i t}\right|}\,dt \le C\cdot e^{-\pi R} \to 0$

Celkem: $(1+e^{ai})J_a = \left(\mathrm{Res}_{i/2} f\right)\cdot 2\pi i = 2\pi i\,\frac{e^{ai/2}}{\pi\cdot e^{i\pi/2}-\pi\cdot e^{-i\pi/2}} = 2\pi i\,\frac{e^{ai/2}}{2\pi i} = e^{ai/2}$

$\Rightarrow \underset{I}{\overset{}{}} J_a = \frac{1}{2\cos\frac{a}{2}}$ pro $a<\pi$. Aby bylo J_{-a} také dobře definováno, musí být i $-a<\pi \Rightarrow |a|<\pi$

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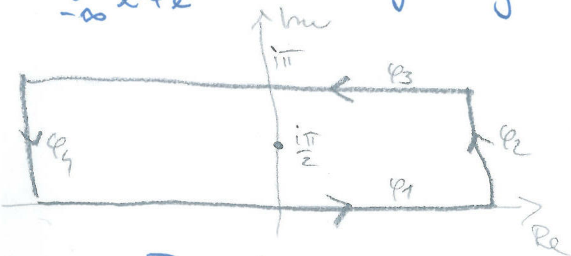
6) $I=\int_0^\infty \frac{\sin^2 ax}{\cosh x}\,dx$, $a>0$ Sudost: $I = \frac12\int_{-\infty}^\infty \frac{\sin^2 ax}{\cosh x}\,dx = \frac12\int_{-\infty}^{\infty}\frac{1-\cos 2ax}{e^x+e^{-x}}\,dx$

$= \frac12\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx - \frac12\int_{-\infty}^{\infty} \mathrm{Re}\,\frac{e^{2iax}}{e^x+e^{-x}}\,dx$ $f(z):=\frac{e^{2iaz}}{e^z+e^{-z}}$. Póly: jako u př. 5), $z_k = \frac{\pi i}{2}+k\pi i$

$J_a := \int_{-\infty}^{\infty} \frac{e^{2iax}}{e^x+e^{-x}}\,dx$. Integrační cesta podobně jako u př. 5),



$\int_{\varphi_1} f\to J_a$, $\int_{\varphi_3} f(z)\,dz = \int_R^{-R}\frac{e^{2ia(i\pi+t)}}{e^{i\pi}\cdot e^t+e^{-i\pi}e^{-t}}\,dt =$

$= e^{-2a\pi}\cdot\int_{-R}^{R}\frac{e^{2iat}}{e^t+e^{-t}}\,dt \to e^{-2a\pi}J_a$

$\left|\int_{\varphi_2} f\right| \le \int_0^\pi \frac{e^{-2at}}{e^R\left|e^{it}+e^{-2R}e^{-it}\right|}\,dt \le C\cdot e^{-R}\to 0$

$\left|\int_{\varphi_4} f\right|\le \int_0^\pi \frac{e^{-2at}}{e^R\left|e^{-it}+e^{-2R}e^{it}\right|}\,dt \le C\cdot e^{-R}\to 0$

Celkem: $J_a\cdot(1+e^{-2a\pi}) = 2\pi i\,\mathrm{Res}_{\frac{i\pi}{2}} f$

$= 2\pi i\,\frac{e^{-a\pi}}{e^{i\pi/2}-e^{-i\pi/2}} = \pi e^{-a\pi}$

$\Rightarrow J_a = \frac{\pi}{2\cosh a\pi}$

To platí i pro $a=0$! $J_0=\frac{1}{2}$

$I = \frac12\left(J_0 - \mathrm{Re}\,J_a\right) = \frac12\left(\frac{\pi}{2} - \frac{\pi}{2\cosh a\pi}\right) = \frac{\pi}{4}\left(1-\frac{1}{\cosh a\pi}\right)$