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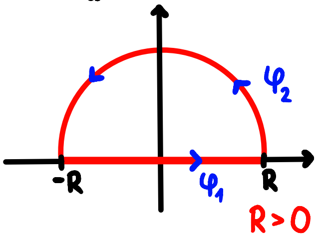
Domáca Úloha IV.

1. (2 body) Spočtěte integrál

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2}\,dx$$

$$I = \int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)}\,dx \;\rightarrow\; f(z) = \frac{z}{(z+2-3i)^2(z+2+3i)^2}$$



$\varphi = \varphi_1 \oplus \varphi_2$ závisí na R

$$\int_{\varphi} f(z)\,dz = 2\pi i \operatorname{Res}(f; -2+3i)$$

$$\parallel$$

$$\underbrace{\int_{\varphi_1} f(z)\,dz}_{\downarrow\, R\to\infty \atop I} + \underbrace{\int_{\varphi_2} f(z)\,dz}_{\searrow\, JL\,* \atop 0}$$

→ Vo vnútri krivky je pól $-2+3i$ násobnosti 2

* → Vidíme, že stupeň čitateľa je 1 zatiaľ čo stupeň menovateľa je 4. Na základe časti a, Jordanovho lemmatu je $M_R = \max_{t\in[0;\pi]} O\left(\frac{1}{R^3}\right)$ a $RM_R \xrightarrow{R\to\infty} 0$

$$\operatorname{Res}(f; -2+3i) = \lim_{z\to -2+3i}\left(\frac{z}{(z+2+3i)^2}\right)' = \lim_{z\to-2+3i} \frac{(z+2+3i)^2 - 2z(z+2+3i)}{(z+2+3i)^4} = \frac{6\cdot 4\cdot i}{6^4} = \frac{i}{54}$$

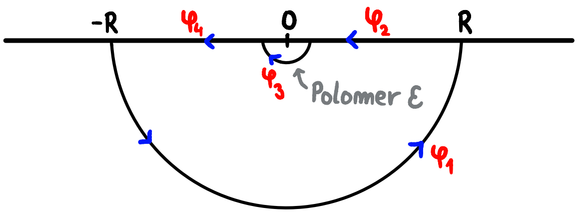
$$I = \lim_{R\to\infty}\left(2\pi i \frac{i}{54}\right) = -\frac{\pi}{27}$$

2. (2 body) Spočtěte integrál

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x}}{x(x^2+1)}\,dx$$

$$I = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x}}{x(x^2+1)}\,dx \;\rightarrow\; f(z) = \frac{e^{-2\pi i z}}{z(z-i)(z+i)}$$

Póly $0, i; -i$ násobnosti 1.



$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \varphi_3 \oplus \varphi_4$$

Pól 0 leží na $\mathbb{R}$, musíme ho obísť.

$$\int_{\varphi} f = 2\pi i \operatorname{Res}(f(z); -i) = \underbrace{\int_{\varphi_2} f + \int_{\varphi_4} f}_{\downarrow\, R\to\infty,\ \varepsilon\to 0 \atop -I} + \int_{\varphi_3} f + \underbrace{\int_{\varphi_1} f}_{JL\ \searrow\, R\to\infty \atop 0}$$

opačný obeh

⊠ → Využitím Jordanovho lemmatu pre záporný polkruh dostávame $b, \alpha = -2\pi < 0$ a $z = Re^{it}$; $t \in [-\pi; 0]$:

$$M_R := \max_{t\in[-\pi;0]}\left(\frac{1}{Re^{it}(R^2e^{2it}+1)}\right) \xrightarrow[R\to\infty]{} 0 \quad \Rightarrow \int_{\varphi_1} f(z)e^{i\alpha z}dz = 0$$

$$\operatorname{Res}\left(\frac{e^{-2\pi i z}}{z(z-i)(z+i)}; -i\right) = \lim_{z\to -i}\frac{e^{-2\pi i z}}{z(z-i)} = \frac{e^{-2\pi}}{-2}$$

$$\operatorname{Res}\left(\frac{e^{-2\pi i z}}{z(z-i)(z+i)}; 0\right) = \frac{1}{-i^2} = 1 \quad \to \int_{\varphi_3} f(z)dz = \lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon = e^{it};\, t\in[-\pi;0]} f \overset{\text{opačný obeh}}{=} -(0+\pi)i1 = -\pi i$$

$$\Rightarrow -\pi i e^{-2\pi} = -I - \pi i \quad \Rightarrow \quad I = -\pi i(1-e^{-2\pi})$$

Vidíme, že integrál vyšiel rýdzo-komplexný ⇒ Integrál vieme "roztrhnúť":

$$\int_{-\infty}^{\infty}\frac{\cos(2\pi x)}{x(x^2+1)}dx = 0 \;;\; \int_{-\infty}^{\infty}\frac{\sin(2\pi x)}{x(x^2+1)}dx = \pi(1-e^{-2\pi})$$