P1-1

P1.1

$$\int_0^{\ln 2} \sqrt{e^x - 1}\, dx \;\left|\; \begin{aligned} e^x &= y \\ e^x dx &= dy \\ dx &= \frac{dy}{y} \\ e^0 &= 1 \\ e^{\ln 2} &= 2 \end{aligned} \right| = \int_1^2 \frac{\sqrt{y-1}}{y}\, dy = \left|\; \begin{aligned} t &= \sqrt{y-1} \\ dt &= \frac{dy}{2\sqrt{y-1}} \Rightarrow \\ 2t\, dt &= dy \\ \text{LB: } t &= 0 \\ \text{UB: } t &= 1 \\ t^2 &= y-1 \end{aligned} \right|$$

$$= 2\int_0^1 \frac{t^2}{t^2+1}\, dt = 2\int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt$$

$$= 2t\Big|_0^1 - 2\arctan t\Big|_0^1 = 2(1-0) - 2\left(\frac{\pi}{4} - 0\right) = \underline{2 - \frac{\pi}{2}}$$

P1.2

$$\int_0^1 \arccos x = \left|\; \begin{aligned} u &= \arccos x & u' &= \frac{-1}{\sqrt{1-x^2}} \\ v' &= 1 & v &= x \end{aligned} \right| = x \arccos x\Big|_0^1$$

$$+ \underbrace{\int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx}_{\left(\sqrt{1-x^2}\right)'} = -(0 \cdot \frac{\pi}{2} - 1 \cdot 0) - \sqrt{1-x^2}\Big|_0^1 = \underline{1}$$

P1.3

$$\int_0^\infty x^{2k-1} e^{-\frac{x^2}{2}} \quad k \in \mathbb{N}$$ (next page)

P1.3

$$\int_0^\infty x^{2k-1} e^{-\frac{x^2}{2}}\,dx = \left|\begin{array}{l} \frac{x^2}{2} = t \\ x\,dx = dt \end{array}\right| \qquad \begin{array}{l} x^2 = 2t \\ x = \sqrt{2}\, t^{1/2} \end{array}$$

$$= \cancel{\int_0^\infty (\sqrt{2t})^{2k-1} e^{-t}}$$

$$= \int_0^\infty \underbrace{x^{2k-2}}_{(x^2)^{k-1}} \underbrace{e^{-\frac{x^2}{2}}}_{e^{-t}} \underbrace{x\,dx}_{dt} = \int_0^\infty (2t)^{k-1} e^{-t}\,dt$$

$$= 2^{k-1} \underbrace{\int_0^\infty t^{k-1} e^{-t}\,dt}_{=\Gamma(k)}$$

$\Gamma(z)$ $\forall z \in \mathbb{C}$ ~~ko~~ $\setminus$ {non-positive integers}

$\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$

$$= 2^{k-1} (k-1)!$$

$$\int_0^\infty t^{k-1} e^{-t}\,dt \overset{1.\,PP}{=} \underbrace{\left[-t^{k-1} e^{-t}\right]_0^\infty}_{0} + (k-1)\int_0^\infty t^{k-2} e^{-t}\,dt$$

$$\overset{2.\,PP}{=} (k-1)\left[-t^{k-2} e^{-t}\right]_0^\infty + (k-1)(k-2)\int_0^\infty t^{k-3} e^{-t}\,dt$$

$$\vdots$$

$$= (k-1)(k-2)\cdots 2\cdot 1 \int_0^\infty t^0 e^{-t}\,dt = (k-1)! \int_0^\infty e^{-t}\,dt$$

$$= (k-1)! \left[-e^{-t}\right]_0^\infty = (k-1)!$$

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Gamma Function

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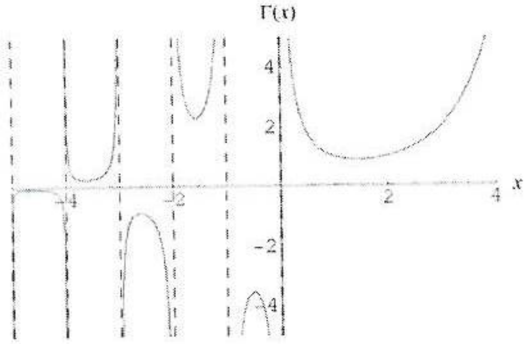


The (complete) gamma function $\Gamma(n)$ is defined to be an extension of the factorial to complex and real number arguments. It is related to the factorial by

$$\Gamma(n) = (n-1)!, \qquad (1)$$

a slightly unfortunate notation due to Legendre which is now universally used instead of Gauss's simpler $\Pi(n) = n!$ (Gauss 1812; Edwards 2001, p. 8).

It is analytic everywhere except at $z = 0, -1, -2, \ldots$, and the residue at $z = -k$ is

$$\operatorname*{Res}_{z=-k} \Gamma(z) = \frac{(-1)^k}{k!}. \qquad (2)$$

There are no points z at which $\Gamma(z) = 0$.

The gamma function is implemented in the Wolfram Language as Gamma[z].

There are a number of notational conventions in common use for indication of a power of a gamma functions. While authors such as Watson (1939) use $\Gamma^n(z)$ (i.e., using a trigonometric function-like convention), it is also common to write $[\Gamma(z)]^n$.

The gamma function can be defined as a definite integral for $\mathbb{R}[z] > 0$ (Euler's integral form)

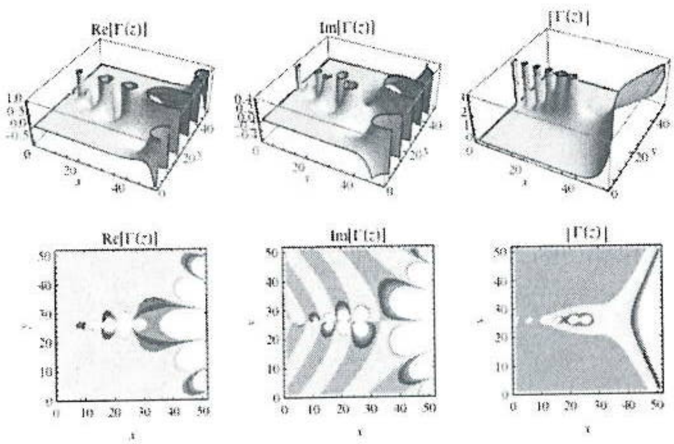
$$\Gamma(z) \equiv \int_0^\infty t^{z-1} e^{-t}\, dt \qquad (3)$$

$$= 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt, \qquad (4)$$

or

$$\Gamma(z) \equiv \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt. \qquad (5)$$

The complete gamma function $\Gamma(x)$ can be generalized to the upper incomplete gamma function $\Gamma(a, x)$ and lower incomplete gamma function $\gamma(a, x)$.



	Min	Max
Re	-5	5
Im	-5	5

Replot

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Plots of the real and imaginary parts of $\Gamma(z)$ in the complex plane are illustrated above.

Integrating equation (3) by parts for a real argument, it can be seen that

gamma function

THINGS TO TRY:
gamma function
gamma function of 2
gamma function of (1/2)

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Stirling's Approximation

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Stirling's approximation gives an approximate value for the factorial function $n!$ or the gamma function $\Gamma(n)$ for $n \gg 1$. The approximation can most simply be derived for n an integer by approximating the sum over the terms of the factorial with an integral, so that

$$\begin{aligned} \ln n! &= \ln 1 + \ln 2 + \ldots + \ln n && (1)\\ &= \sum_{k=1}^{n} \ln k && (2)\\ &\approx \int_1^n \ln x \, dx && (3)\\ &= [x \ln x - x]_1^n && (4)\\ &= n \ln n - n + 1 && (5)\\ &\approx n \ln n - n. && (6) \end{aligned}$$

The equation can also be derived using the integral definition of the factorial,

$$n! = \int_0^\infty e^{-x} x^n \, dx. \quad (7)$$

Note that the derivative of the logarithm of the integrand can be written

$$\frac{d}{dx} \ln(e^{-x} x^n) = \frac{d}{dx}(n \ln x - x) = \frac{n}{x} - 1. \quad (8)$$

The integrand is sharply peaked with the contribution important only near $x = n$. Therefore, let $x \equiv n + \xi$ where $\xi \ll n$, and write

$$\begin{aligned} \ln(x^n e^{-x}) &= n \ln x - x && (9)\\ &= n \ln(n + \xi) - (n + \xi). && (10) \end{aligned}$$

Now,

$$\begin{aligned} \ln(n + \xi) &= \ln\left[n\left(1 + \frac{\xi}{n}\right)\right] && (11)\\ &= \ln n + \ln\left(1 + \frac{\xi}{n}\right) && (12)\\ &= \ln n + \frac{\xi}{n} - \frac{1}{2}\frac{\xi^2}{n^2} + \ldots, && (13) \end{aligned}$$

so

$$\begin{aligned} \ln(x^n e^{-x}) &= n \ln(n + \xi) - (n + \xi) && (14)\\ &= n \ln n + \xi - \frac{1}{2}\frac{\xi^2}{n} - n - \xi + \ldots && (15)\\ &= n \ln n - n - \frac{\xi^2}{2n} + \ldots. && (16) \end{aligned}$$

Taking the exponential of each side then gives

$$\begin{aligned} x^n e^{-x} &\approx e^{n \ln n} e^{-n} e^{-\xi^2/2n} && (17)\\ &= n^n e^{-n} e^{-\xi^2/2n}. && (18) \end{aligned}$$

Plugging into the integral expression for $n!$ then gives

$$\begin{aligned} n! &\approx \int_{-n}^{\infty} n^n e^{-n} e^{-\xi^2/2n} \, d\xi && (19)\\ &\approx n^n e^{-n} \int_{-\infty}^{\infty} e^{-\xi^2/2n} \, d\xi. && (20) \end{aligned}$$

Evaluating the integral gives

$$\begin{aligned} n! &\approx n^n e^{-n} \sqrt{2\pi n} && (21)\\ &= \sqrt{2\pi}\, n^{n+1/2} e^{-n} && (22) \end{aligned}$$

(Wells 1986, p. 45). Taking the logarithm of both sides then gives

$$\begin{aligned} \ln n! &\approx n \ln n - n + \tfrac{1}{2} \ln(2\pi n) && (23)\\ &= \left(n + \tfrac{1}{2}\right) \ln n - n + \tfrac{1}{2} \ln(2\pi). && (24) \end{aligned}$$

This is Stirling's series with only the first term retained and, for large n, it reduces to Stirling's approximation

$$\ln n! \approx n \ln n - n. \quad (25)$$

Taking successive terms of $\lfloor n^n / n! \rfloor$, where $\lfloor x \rfloor$ is the floor function, gives the sequence 1, 2, 4, 10, 26, 64, 163, 416, 1067, 2755, ... (OEIS A055775).

Stirling's approximation can be extended to the double inequality

$$\sqrt{2\pi}\, n^{n+1/2} e^{-n+1/(12n+1)} < n! < \sqrt{2\pi}\, n^{n+1/2} e^{-n+1/(12n)} \quad (26)$$

(Robbins 1955, Feller 1968).

Gosper has noted that a better approximation to $n!$ (i.e., one which approximates the terms in Stirling's series instead of truncating them) is given by

P1.4 $\displaystyle\int_0^{4\pi} \frac{1}{1+\sin^2 x}\,dx = \left| \boxed{y = \tan x} \; x \neq \frac{\pi}{2} + k\pi \quad dy = \frac{dx}{\cos^2 x} \right|$ (P1.2)

$\underbrace{\frac{1}{1+\sin^2 x}}$ vadí v sinu a cosinu

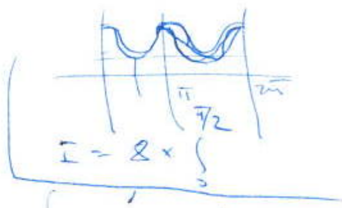
$$\sin^2 x = \frac{\sin^2 x}{\sin^2 x + \cos^2 x} = \frac{\tan^2 x}{1+\tan^2 x} = \frac{y^2}{1+y^2} \qquad \frac{dy}{1+y^2} = dx$$

$$= 4 \times \underbrace{\int_0^{\pi} \frac{1}{1+\sin^2 x}}_{\frac{\pi}{\sqrt{2}}}$$

$$4\,\frac{\pi}{\sqrt{2}} = \underline{\underline{2\sqrt{2}\,\pi}}$$

$$\cos^2 x = \frac{\cos^2 x}{\cos^2 x + \sin^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

$$\frac{1}{1+\sin^2 x}$$



$$I = 8 \times \int_0$$

$$= \int \frac{1}{1+\frac{y^2}{1+y^2}}\,\frac{dy}{1+y^2} = \int \frac{1}{\frac{1+2y^2}{\cancel{1+y^2}}}\,\frac{dy}{\cancel{1+y^2}} = \int \frac{1}{1+2y^2}\,dy$$

$$\left| \begin{array}{l} \sqrt{2}\,y = z \\ \sqrt{2}\,dy = dz \end{array} \right| = \frac{1}{\sqrt{2}} \int \frac{1}{1+z^2}\,dz = \frac{1}{\sqrt{2}} \arctan z$$

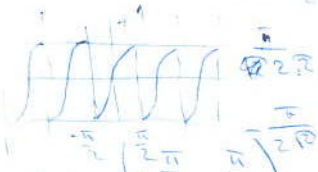
$$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\,y) = \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x) = F(x)$$

∘ Tedy: $\displaystyle\int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx = \underset{0}{F(\pi)} - \underset{0}{F(0)} = 0\ ?$

Newton-Leibnitz

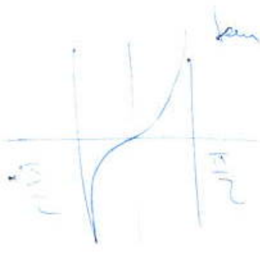
Ale $f(x)$ spojitá > 0 na $[0,\pi]$

$$f(x) = \frac{1}{1+\sin^2 x}$$



$$F(x) = \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x) \text{ na } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

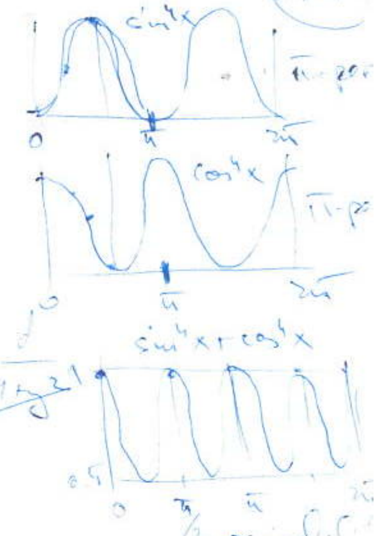
$$\lim_{x\to\frac{\pi}{2}\pm} F(x) = \lim_{x\to\frac{\pi}{2}\pm} \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan x) = \mp \frac{\pi}{\sqrt{2}\,2}$$



$$\tilde{F}(x) \text{ na } [0,\pi] \begin{cases} F(x) & x \in [0, \frac{\pi}{2}) \\ \frac{\pi}{2\sqrt{2}} & x = \frac{\pi}{2} \\ F(x) + \frac{\pi}{2} & x \in (\frac{\pi}{2}, \pi] \end{cases}$$

P 1.5 $\displaystyle\int_0^{2\pi} \frac{1}{\sin^4 x+\cos^4 x}\,dx = \underline{\underline{2\sqrt{2}\,\pi}}$ $\boxed{y=\tan x}$

P1-3



$$= 4\cdot\int_0^{\pi/2}\frac{1}{\sin^4 x+\cos^4 x}\,dx$$

$$I_4 = \int_0^{\infty}\frac{1}{\frac{y^4}{(1+y^2)^2}+\frac{1}{(1+y^2)^2}}\,\frac{dy}{1+y^2} \qquad \frac{1}{\frac{1+y^4}{(1+y^2)^2}} \qquad \frac{dy}{1+y^2}$$

$$= \int_0^{\infty}\frac{1+y^2}{1+y^4}\,dy$$

$$y = \pm\frac{\sqrt2}{2}(\pm1\pm i)$$

$$\left(y-\frac{\sqrt2}{2}(1+i)\right)\left(y-\frac{\sqrt2}{2}(1-i)\right)\left(y+\frac{\sqrt2}{2}(1+i)\right)\left(y+\frac{\sqrt2}{2}(1-i)\right)$$

$$= \left(y^2 - y\frac{\sqrt2}{2}(1+i+1-i)+\frac{2}{4}(1+i)(1-i)\right)(\quad)(\quad)$$

$$= \left(y^2-\sqrt2\,y+1\right)\left(y^2+\sqrt2\,y+1\right)$$

$$= \frac12\int\left(\frac{1}{y^2+\sqrt2\,y+1}+\frac{1}{y^2-\sqrt2\,y+1}\right)dy$$

$$= \frac12\int\frac{dy}{\left(y+\frac{1}{\sqrt2}\right)^2+\frac12} + \frac12\int\frac{dy}{\left(y-\frac{1}{\sqrt2}\right)^2+\frac12}$$

$$= \int\frac{dy}{\left(\sqrt2\left(y+\frac{1}{\sqrt2}\right)\right)^2+1} + \int\frac{dy}{\left(\sqrt2\left(y-\frac{1}{\sqrt2}\right)\right)^2+1}$$

$\sqrt2\left(y+\frac{1}{\sqrt2}\right) = z$

$\sqrt2\,dy = dz$

$dy = \frac{dz}{\sqrt2}$

$\sqrt2\left(y-\frac{1}{\sqrt2}\right) = \tilde z$

$$= \frac{1}{\sqrt2}\left(\int\frac{dz}{z^2+1}+\int\frac{d\tilde z}{\tilde z^2+1}\right)$$

$$= \frac{1}{\sqrt2}\left(\arctan(1+z)+\arctan(1+\tilde z)\right)$$

$$= \frac{1}{\sqrt2}\left[\arctan\left(\sqrt2\,y+1\right)+\arctan\left(\sqrt2\,y-1\right)\right]$$

$$= \frac{1}{\sqrt2}\left[\arctan\left(\sqrt2\tan x+1\right)+\arctan\left(\sqrt2\tan x-1\right)\right]\Bigg|_0^{\pi/2} = \frac{1}{\sqrt2}\left[\frac{\pi}{2}-\arctan(1)+\frac{\pi}{2}-\arctan(-1)\right] = \frac{\pi}{\sqrt2}$$

$$I_4 = \int_0^{\pi/2}\frac{1}{\sin^4 x+\cos^4 x}\,dx = \dots \underline{\frac{\pi}{\sqrt2}}$$

P1-4

P1.6 $\int_2^\infty \frac{1}{x^2} dx = -\frac{1}{x}\Big|_2^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$

P1.7 $\int_0^\infty e^{-3x} dx = \left(-\frac{e^{-3x}}{3}\right)\Big|_0^\infty = -0 + \frac{1}{3} = \frac{1}{3}$

$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} \overset{l'H}{=} \frac{\frac{1}{x}}{-2x^{-3}} = -\frac{x^2}{2}$

P1.8 $\int_0^1 x \ln x = \begin{vmatrix} u = \ln x & u' = \frac{1}{x} \\ v' = x & v = \frac{x^2}{2} \end{vmatrix} = \left(\frac{1}{2} x^2 \ln x\right)\Big|_0^1 - \frac{1}{2}\int_0^1 x\, dx$

$= \left(-\frac{x^2}{4}\right)\Big|_0^1 = -\frac{1}{4}$

$e^{ibx} = \cos bx + i \sin bx$

$\cos bx = \mathrm{Re}(e^{ibx})$

P1.9 $\int_0^\infty e^{-ax} \cos(bx)\, dx = \mathrm{Re} \int_0^\infty e^{(-a+ib)x} dx$

$= \mathrm{Re}\left(\frac{e^{(-a+ib)x}}{-a+ib}\right) = \mathrm{Re}\left(e^{-ax} \frac{e^{ibx}}{-a+ib}\right)$

$a > 0$

$= \mathrm{Re}\left(e^{-ax} \frac{\cos bx + i \sin bx}{-a+ib}\right) = \mathrm{Re}\left(e^{-ax} \frac{(\cos bx + i \sin bx)(-a-ib)}{a^2 + b^2}\right)$

$= \frac{e^{-ax}}{a^2+b^2}\left(-a \cos bx + b \sin bx\right)\Big|_0^\infty = \frac{0-(a)}{a^2+b^2} \quad (+?)$

$= \frac{a}{a^2+b^2}$

P1.10 $\int_0^{\pi/2} \tan x\, dx$

$\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx \begin{vmatrix} t = \cos x \\ dt = -\sin x\, dx \end{vmatrix} = -\int \frac{dt}{t}$

$= -\ln t = -\ln(\cos x)$

$\Rightarrow = -\left(\ln(\cos x)\right)\Big|_0^{\pi/2} \ldots$ nekonvergira

P1.11 $\displaystyle\int_0^{\pi} \ln(1-2\alpha\cos x+\alpha^2)\,dx$ (M-5)

$\cos(\pi - y) = -\cos y$

$y = \pi - x$

$dy = -dx$

$x=0 \to y=\pi$

$x=\pi \to y=0$

$$-\int_{\pi}^{0} \ln(1+2\alpha\cos y+\alpha^2)\,dy = \int_0^{\pi} \ln(1+2\alpha\cos y+\alpha^2)\,dy$$

$I(a) = I(-a)$

$$I(a)+I(-a) = \int_0^{\pi}\left[\ln(1+2\alpha\cos x+\alpha^2) + \ln(1-2\alpha\cos x+\alpha^2)\right]dx$$

$$= \int_0^{\pi} \ln\left[(1+2a\cos x+a^2)(1-2a\cos x+a^2)\right]dx$$

$$= \int_0^{\pi} \ln\left[(1+a^2)^2-(2a\cos x)^2\right]dx$$

$\cos(2x) = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$1+\cos 2x = 2\cos^2 x$

$$= \int_0^{\pi} \ln\left[1+\cancel{2a^2}+a^4-2a^2(\cancel{1}+\cos 2x)\right]dx$$

$$= \int_0^{\pi} \ln\left[1-2a^2\cos 2x+a^4\right]dx$$

$$= \left|\begin{array}{l} y=2x \\ dy = 2dx \\ y = 0..2\pi \end{array}\right| = \frac{1}{2}\int_0^{2\pi} \ln(1-2a^2\cos y+a^4)\,dy$$

$$= \frac{1}{2}\int_0^{\pi} \ln(1-2a^2\cos y+a^4)\,dy + \frac{1}{2}\int_{\pi}^{2\pi} \ln(1-2a^2\cos y+a^4)\,dy$$

$$= \frac{1}{2}\int_0^{\pi} \ln(1-2a^2\cos y+a^4)\,dy + \left|\begin{array}{l} x=2\pi-y \\ dx=-dy \\ x=\pi..0 \end{array}\right| -\int_{\pi}^{0} \ln(1-2a^2\cos x+a^4)\,dx$$

$$= \frac{1}{2}I(a^2) + \frac{1}{2}I(a^2) = I(a^2)$$

$2I(a) = I(a^2)$

$a=0$ $\quad$ ~~$2I(a)$~~ $2I(0) = I(0) \Rightarrow I(0) = 0$

$a=1$ $\quad$ $2I(1) = I(1) \Rightarrow I(1) = 0$

For $0 < a < 1$

$$I(a) = \frac{1}{2} I(a^2) = \frac{1}{2}\,\frac{1}{2} I(a^4) = \dots \frac{1}{2^n} I(a^{2n})$$

$$n \to \infty \qquad I(a) = \lim_{n\to\infty} \frac{1}{2^n} I(a^{2n}) = 0$$

For $a > 1$

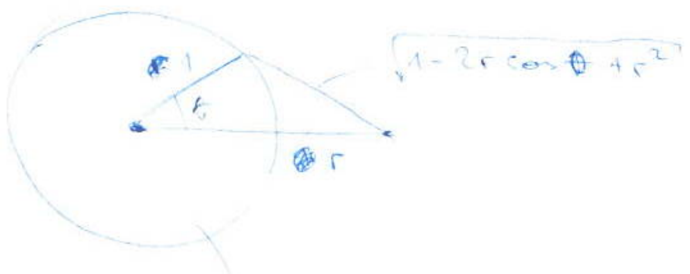
$$I(a) = \int_0^\pi \ln\left\{a^2\left(\frac{1}{a^2} + \frac{2}{a}\cos x + 1\right)\right\}dx$$

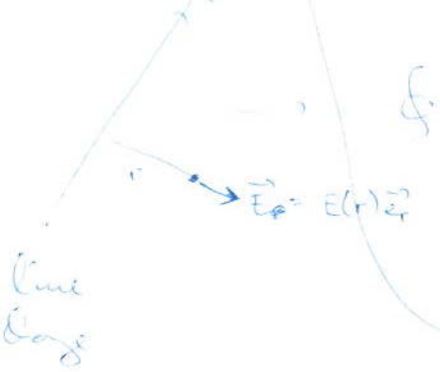
$$= \int_0^\pi \ln a^2\, dx + \underbrace{\int_0^\pi \ln\left(1 + \frac{2}{a}\cos x + \frac{1}{a^2}\right)dx}_{I(1/a)\, =\, 0}$$

$$= 2\pi \ln a$$

$$\to I(a) = \begin{cases} 0 & \text{if } |a| \le 1 \\ 2\pi \ln|a| & |a| > 1 \end{cases}$$

Fyzikální motivace





$$\to \oint \vec{E}\cdot d\vec{S} = 2\pi r E(r) l = \int \frac{\rho}{\varepsilon_0}\, dV = \frac{\lambda l}{\varepsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{e}_r$$

$$\vec{E} = -\nabla V$$

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln r$$

line $\to$ cylinder: $\quad r \to \ln\sqrt{1 - 2r\cos\theta + r^2} = \frac{1}{2}\ln(1 - 2r\cos\theta + r^2)$

Riemann sum

P1-7

$\int \ln(1-2a\cos x+a^2)\,dx$

$(a-z)(a-z^*)$

$[a-(b+ic)][a-(b-ic)]$

$= a^2 - a(b+ic+b-ic) + (b+ic)(b-ic)$

$\frac{\pi}{n}\left[\ln\left(1-2a\cos\frac{\pi}{n}+a^2\right)+\ln\left(1-2a\cos\frac{2\pi}{n}+a^2\right)+\dots+\ln\left(1-2a\cos\frac{(n-1)\pi}{n}+a^2\right)\right]$

$= \frac{\pi}{n}\ln\prod_{k=1}^{n}\left(1-2a\cos\left(\frac{k\pi}{n}\right)+a^2\right)$

$z = b+ic$

$(a-z)(a-z^*) = a^2 - a(z+z^*) + zz^*$

$= a^2 - a(b+\cancel{ic}+b-\cancel{ic}) + (b+ic)(b+ic)$

$= a^2 - 2ab + b^2+c^2$

$\cos(\;)\quad \cos^2+\sin^2$

$z = \cos + i\sin$

$= \frac{\pi}{n}\ln\prod_{k=1}^{n}\left[a-\left(\cos\frac{k\pi}{n}+i\sin\frac{k\pi}{n}\right)\right]\left[a-\left(\cos\left(\frac{k\pi}{n}\right)-i\sin\frac{k\pi}{n}\right)\right](a+1)(a-1)$

$\overline{(a+1)(a-1)}$

~~$z^n - a^n$~~ a ~~$(a-1)(a+1)$~~

$a^{2n}-1$

$a^{2n}=1$

$= \frac{\pi}{n}\ln\frac{1-a^{2n}}{1-a^2} = \frac{\pi}{n}\ln\frac{1-(a^2)^n}{1-a^2}$

$|a|<1 \quad \lim_{n\to\infty}\frac{\pi}{n}\ln\frac{1-a^{2n}}{1-a^2} = 0$

$|a|>1 \quad \lim_{n\to\infty}\frac{\pi}{n}\ln\frac{1-a^{2n}}{1-a^2}$ ~~$= \lim \frac{\pi}{n}\ln\frac{a^{2n}-1}{a^2-1}$~~

$= \lim\left[\frac{\pi}{n}\ln(a^{2n}-1)-\ln(a^2-1)\right]$

$= \lim_{n\to\infty}\left(\frac{\pi}{n}2n\ln a - \frac{\pi}{n}\ln(a^2-1)\right)$

$= 2\pi\ln a$

P 1.12 $\int_0^\infty x^p\,dx \quad p\in\mathbb{R}$

$p\neq -1 \quad \int x^p = \frac{x^{p+1}}{p+1}$ $\qquad \int_0^\infty x^p = \lim_{x\to\infty} \frac{x^{p+1}}{p+1} - \lim_{x\to 0} \frac{x^{p+1}}{p+1}$

$p=-1 \quad \int \frac{1}{x} = \ln x$ $\qquad \int_0^\infty \frac{1}{x} = \lim_{x\to\infty} \ln x - \lim_{x\to 0+} \ln x = \infty - (-\infty)$

$p>-1 \ : \lim_{x\to\infty} \frac{x^\alpha}{\alpha} - \lim_{x\to 0+} \frac{x^\alpha}{\alpha} = \text{"}\infty - 0\text{"}$

$p<-1 \ : -\lim_{x\to\infty} \frac{1}{\beta x^\beta} + \lim_{x\to 0+} \frac{1}{\beta x^\beta} = \text{"}0 + \infty\text{"}$

Integrál nikdy nekonverguje.

P 1.13 $\int_1^\infty x^p\,dx \quad p\in\mathbb{R}$: $p<-1 \quad \int_1^\infty x^p\,dx = -\frac{1}{p+1}$ ✓

konverguje pro

P 1.14 $\int_0^{10} x^p\,dx \quad p\in\mathbb{R}$: $p>-1 \quad \int_0^{10} x^p\,dx = \frac{10^{p+1}}{p+1}$ ✓

P 1.15 $\int_0^\infty \frac{x^{3/2}}{1+x^2}\,dx = \int_0^a \frac{x^{3/2}}{1+x^2}\,dx + \int_a^\infty \frac{x^{3/2}}{1+x^2}\,dx$

See Kopáček, p 155 vičení

$x\to 0: \quad \frac{x^{3/2}}{1+x^2} \sim x^{3/2}$ není

$x\to\infty: \quad \frac{x^{3/2}}{1+x^2} \sim x^{-1/2} \quad \lim \frac{\frac{x^{3/2}}{1+x^2}}{x^{-1/2}} = \lim \frac{x^2}{1+x^2} = \lim_{x\to 0} \frac{1}{1+\frac{1}{x^2}} = 1$

f, g spojité nezáporné funkce na $\langle a, +\infty)$

$\lim_{x\to 0+} \frac{f}{g}$ = vlastní nenulová → oba konvergují nebo oba nekonvergují

$\int_a^\infty f(x)\,dx \quad \int_a^\infty g(x)\,dx$

→ x^p kde $p=-\frac{1}{2}$ → nekonverguje

(K) pouze pro $p<-1$

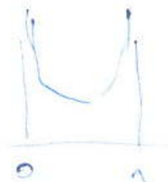
(P1-9)

$$\int_0^1 \frac{x^{3/2}}{1+x^2}dx \le \int_0^1 \frac{x^{3/2}}{x^2}dx = \int_0^1 x^{-1/2}dx = \left[2x^{1/2}\right]_0^1 = 2$$

$$\int_1^\infty \frac{x^{3/2}}{1+x^2} \ge \int_1^\infty \frac{x^{3/2}}{2x^2}dx = 2\int_1^\infty x^{-1/2} = \left[4x^{1/2}\right]_1^\infty = \infty$$

Nekonvergira (+∞ in the problem)

P1.16 $\int_0^1 \frac{1}{\sqrt{x(1-x^2)}}dx$



$f(x) > 0$ na $(0,1)$

$x\to 0_+$: $\frac{1}{\sqrt{x(1-x^2)}} \sim x^{-1/2}$ $\quad p > -1$ OK?

$x\to 1_-$: $\frac{1}{\sqrt{x(1-x^2)}} \sim \frac{1}{\sqrt{x(1+x)(1-x)}} \sim \frac{1}{\sqrt{1-x}}$

$$\int_0^{1/2}\frac{dx}{\sqrt{x(1-x^2)}} \le \frac{[illegible]}{2}\int_0^{1/2}\frac{1}{\sqrt{x}} = [illegible]\sqrt{x}\Big|_0^{1/2}$$

$$\int_{1/2}^1 \frac{dx}{\sqrt{x(1-x^2)}} \le \sqrt{2}\int_{1/2}^1 \frac{dx}{\sqrt{1-x^2}} = \sqrt{2}\left[\arcsin x\right]_{1/2}^{1} = \sqrt{2}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \sqrt{2}\,\frac{\pi}{3}$$

P1.17 $\int_0^2 \frac{1}{\ln x}dx \;\left|\begin{array}{l} x = e^t \\ dx = e^t dt \\ t\to -\infty \end{array}\right| = \int_{-\infty}^{\ln 2}\frac{e^t}{t}dt$

$$\int_0^{\ln 2}\frac{e^t}{t} \ge \int_0^{\ln 2}\frac{1}{t} = \left[\ln t\right]_0^{\ln 2} = \ln\ln 2 - (-\infty) = \infty$$

$$\int_{-\infty}^0 \frac{e^t}{t} = \int_{-\infty}^{-1}\frac{e^t}{t}dt + \underbrace{\int_{-1}^0 \frac{e^t}{t}dt}_{\le 0} \le \int_{-1}^0 \frac{e^t}{t}dt \le e^{-1}\int_{-1}^0\frac{1}{t} = \left[e^{-1}\ln|t|\right]_{-1}^0 = -\infty$$

→ not converges ✗

P1.18

$$\int_0^{\frac{\pi}{2}} \frac{\ln \sin x}{x^p} \, dx \qquad p \in \mathbb{R}$$

$p=0$ ✓

$p=1$ ✗

$0<p<1$ ✓

$$\lim_{x \to 0+} \frac{\ln \sin x}{x^p} = \begin{cases} p \geq 0 & -\infty \quad \text{pro } \underline{p \geq 0} \\ p<0 & \overset{l'H}{=} \lim_{x\to 0+} \dfrac{\frac{\cos x}{\sin x}}{p x^{p-1}} = \dfrac{1}{p} \lim \cos x \dfrac{x^{1-p}}{\sin x} \end{cases}$$

$$= \frac{1}{p} \lim_{x \to 0+} \underbrace{\cos x}_{1} \underbrace{\frac{x}{\sin x}}_{1} \underbrace{x^{-p}}_{0} = 0 \quad \text{pro } \underline{p<0}$$

$\underline{p \geq 0}$

$$\int_0^{\pi/2} \frac{\ln \sin x}{x^p} \, dx \leq \int_0^{\pi/2} \frac{\ln x}{x^p} \, dx \quad \left| \begin{array}{ll} u = \ln x & u' = \frac{1}{x} \\ v' = x^{-p} & v = \frac{x^{1-p}}{1-p} \end{array} \right|$$

$$= \left[\ln x \, \frac{x^{1-p}}{1-p} \right]_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \frac{x^{-p}}{1-p} \, dx$$

$$= \left[\ln x \, \frac{x^{1-p}}{1-p} - \frac{x^{1-p}}{(1-p)^2} \right]_0^{\pi/2} = \frac{1}{(1-p)^2} \left[x^{1-p} \left((1-p) \ln x - 1 \right) \right]_0^{\pi/2}$$

- $p > 1$: $\longrightarrow -\infty$
- $p = 1$:

P1.19

$$\int_0^\infty \frac{\operatorname{arctan} x}{x^{3/2}}\, dx$$

$$\frac{\operatorname{arctan} x}{x^{3/2}} \geq 0 \text{ na } [0,\infty)$$

$$\int_0^1 \frac{\operatorname{arctan} x}{x^{3/2}}\, dx \leq \int_0^1 \frac{x}{x^{3/2}}\, dx = \int_0^1 \frac{dx}{\sqrt{x}} = 2\left[\sqrt{x}\right]_0^1 = 2$$

$\operatorname{arctan} x \leq x$

$$\int_1^\infty \frac{\operatorname{arctan} x}{x^{3/2}}\, dx \leq \frac{\pi}{2} \int_1^\infty \frac{1}{x^{3/2}}\, dx = \frac{\pi}{2} \int_1^\infty x^{-3/2}\, dx = \frac{\pi}{2} \left[\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_1^\infty$$

$\operatorname{arctan} x < \frac{\pi}{2}$

$$= \pi \left[\frac{1}{\sqrt{x}}\right]_\infty^1 = \pi$$

K

Věta (7.6.5) Srovnávací kritérium pro konvergenci Newtonova integrálu

Nechť $-\infty < a < b \leq +\infty$

- Jestliže $f \in C([a,b))$, $g \geq 0$ newtonovsky integrovatelná na (a,b) a $f(x) = O(g(x))$ pro $x \to b_-$ $\Rightarrow$ pak i f je newtonovsky integrovatelná na (a,b)
- Jestliže $f, g \in C([a,b))$ nezáporné, $f(x) = O(g(x))$ a $g(x) = O(f(x))$ pro $x \to b_-$ $\Rightarrow$ pak f newt. integr. na (a,b) $\Leftrightarrow$ g newt. integr. na (a,b)
- Speciálně pokud $f, g, h \in C([a,b))$ nezáporné, $f(x) = O(g(x))$ a $g(x) = O(f(x))$ pro $x \to b_-$, pak fh je newtonovsky integr. na (a,b) $\Leftrightarrow$ gh newt. integr. na (a,b)

Příklad $(N)\int_0^1 \underbrace{\frac{e^x - 1}{\sqrt{x}\sin x}}_{= f} dx$ $\quad g = \frac{1}{\sqrt{x}}$ $\quad \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim \frac{e^x - 1}{\sin x} = 1$ $\quad (N)\int_0^1 \frac{1}{\sqrt{x}} dx = \left[2\sqrt{x}\right]_0^1 = 2 \in \mathbb{R}$

Příklad $(N)\int_0^1 \ln x \sin\frac{1}{x} dx$ $\quad g = |\ln x| = -\ln x$ $\quad |f| \leq g$ na $(0,1)$

$(N)\int_0^1 -\ln x \, dx = -\left[x\ln x - x\right]_0^1 = 1 \Rightarrow \int_0^1 \ln x \sin\frac{1}{x} dx$ existuje

Příklad $\underbrace{(N)\int_0^{+\infty} \operatorname{arctan} x \, dx}$ $\quad g(x) = 1$ ... není newt. integr., zároveň $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\pi}{2} \to$

$\Rightarrow$ neexistuje

Věta (7.6.11) Abel-Dirichlet kritérium

Nechť $F, f, g \in C([a,b))$, $F' = f$ na (a,b) a g monotónní na (a,b)

- (Dirichlet) Je-li F omezená na (a,b) a $\lim_{x \to b_-} g(x) = 0$ $\Rightarrow$ fg newt. integr. na (a,b)
- (Abel) Je-li f newt. integr. na (a,b) a g omezená na (a,b) $\Rightarrow$ fg newt. integr. na (a,b)

Příklad. $\int_{a}^{\infty} \frac{\sin x}{\sqrt{x}} dx$ $\quad \left.\begin{array}{l} f = \sin x \text{ ... } F = -\cos x \text{ omezená} \\ g = \frac{1}{\sqrt{x}} \text{ ... monotónní, } \lim_{x \to +\infty} \frac{1}{\sqrt{x}} = 0 \end{array}\right\} \to \int_0^{\infty} \frac{\sin x}{\sqrt{x}} dx$ existuje

$\int_{2\pi}^{\infty} \frac{\sin^2 x}{x} dx = \int_{2\pi}^{\infty} \left(\frac{1}{2x} - \frac{\cos(2x)}{2x}\right) dx$

$\quad$ k dle Dirichleta

$f = \sin^2 x$ ... F není omezená $\left(= \frac{x}{2} - \frac{1}{4}\sin(2x)\right)$