Sada příkladů 2/12

Funkce více proměnných

Lokální extrémy funkcí více proměnných

Hledejte lokální extrémy následujících funkcí

1. $x^2 + y^2$; $x^2 - y^2$; $-x^2 - y^2$
2. $x^4 + y^4 - x^2 - 2xy - y^2$
3. $(x^2 + y^2)\mathrm{e}^{-(x^2+y^2)}$
4. $(2x^2 - xy + y^2/3 - 5x + 5y/3 + 10/3)\mathrm{e}^{x+y}$
5. $$f(x) = \begin{cases} xy\ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$
6. $x + y + 4\cos x \cos y$
7. $\sin x + \cos y + \cos(x - y)$ na intervalu $\left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right)$
8. $x - 2y + \ln(\sqrt{x^2 + y^2}) + 3\operatorname{arctg}\frac{y}{x},\ x \neq 0$
9. $x^2 + y^2 + z^2 + 2x + 4y - 6z$
10. $(ax + by + cz)\mathrm{e}^{-x^2-y^2-z^2}$.

Implicitní funkce

11. Dokažte, že existuje okolí V bodu $(1, 1)$ takové, že množina
$$\{(x, y); x^3 + y^3 - 2xy = 0\} \cap V$$
je grafem nějaké funkce, která je třídy C^2 na nějakém okolí bodu 1. Spočtěte $f'(1)$ a $f''(1)$.

12. Dokažte, že existuje okolí V bodu $(3,-2,2)$ takové, že množina

$$\{(x,y,z); z^3 - xz + y = 0\} \cap V$$

je grafem nějaké funkce, která je třídy C^2 na nějakém okolí bodu $(3,-2)$. Spočtěte $\frac{\partial^2 z}{\partial y^2}(3,-2)$.

13. Spočtěte parciální derivace 2. řádu funkce implicitně zadané vztahem $x + y + z = \mathrm{e}^{-(x+y+z)}$.

14. Nalezněte první a druhý diferenciál funkce dané vztahem $z = x + \operatorname{arctg} \frac{y}{z-x}$.

15. Jsou-li $x = f(y,z)$, $y = g(x,z)$, $z = h(x,y)$ implicitně zadány vztahem $F(x,y,z) = 0$, ukažte, že $f_y g_z h_x = -1$.

16. Napište $\mathrm{d}u$ a $\mathrm{d}v$, je-li $u + v = x + y$, $\frac{\sin u}{\sin v} = \frac{x}{y}$.

17. Hledejte lokální extrémy funkce $z = z(x,y)$, dané implicitně vztahem

$$(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2 - z^2)\,.$$

Lokální extrémy funkcí více proměnných

$f(x_1, \dots, x_n)$

→ parciální derivace neexistují (všechny) } podezřelé body
→ parciální derivace ∃ a rovnají se nule } (z extrému)

← „stacionární body"

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1} &= 0 \\ &\vdots \\ \frac{\partial f}{\partial x_n} &= 0 \end{aligned}\right\} \text{ n rovnic pro n neznámých: } x_1, \dots, x_n$$

Hessián

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

dosadíme podezřelý bod $(x_1, \dots, x_n)$

- Definitnost kvadratické formy
  - převod na □
  - diagonalizace
  - Sylvester

→ matice
- pozitivně definitní … lok. minimum
- negativně definitní … lok. maximum
- indefinitní … sedlový bod
- semidefinitní … nevíme

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Subdeterminanty podle diagonály:

$$\left.\begin{aligned} & a_{11} > 0 \\ & a_{11}a_{22} - a_{12}a_{21} > 0 \\ & \vdots \\ & \begin{vmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{vmatrix} > 0 \end{aligned}\right\} \Rightarrow \text{pozitivně definitní}$$

semidefinitní
→ někde nastane rovnost (=0)

indefinitní
→ jiná kombinace > & < 0

$$\left.\begin{aligned} & a_{11} < 0 \\ & a_{11}a_{22} - a_{12}a_{21} > 0 \\ & \vdots \\ & \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \begin{cases} > 0 \text{ sudé} \\ < 0 \text{ liché} \end{cases} \end{aligned}\right\} \Rightarrow \text{negativně definitní}$$

$\boxed{\mathbb{R}^2}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ → symetrická $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ … $\det A = ac - b^2 > 0$ → definitní

$a > 0 \Rightarrow$ pos. def.
$a < 0 \Rightarrow$ neg. def.

Vázané extrémy

$f(x_1, \dots, x_n)$

vazba $g(x_1, \dots, x_n) = 0$

1) $x_1 = \tilde{g}(x_2, \dots, x_n)$ ... $\tilde{f}(x_2, \dots, x_n) = f(\tilde{g}(x_2, \dots, x_n), x_2, \dots, x_n)$

2) Lagrangeovy multiplikátory

$\exists \lambda \in \mathbb{R}: \quad \nabla f = \lambda \nabla g$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} = \lambda \frac{\partial g}{\partial x_n} \end{array}\right\} n \text{ rovnic}$$

$g(x_1, \dots, x_n) = 0$ .. 1 rovnice

$(n+1)$ rovnic pro $n+1$ neznámých $x_1, \dots, x_n, \lambda$

$F(x_1, \dots, x_n, \lambda) = f - \lambda g$

Globální extrémy

- f spojitá na M, M omezená a uzavřená $\Rightarrow$ f nabývá max, min
- — " — , $M = \mathbb{R}^N$, $\lim_{\|x\| \to \infty} f(x) = \infty$ $\Rightarrow$ f nabývá min (simprovax)
- — " — , $M = \mathbb{R}^N$, $\lim_{\|x\| \to \infty} f(x) = 0$, $\exists x_0 : f(x_0) < 0$ $\Rightarrow$ f nabývá min

Vázané extrémy – 2 vazby ($\to$ n vazeb)

$f(x_1, \dots, x_n)$

$g_1(x_1, \dots, x_n) = 0$ ... $\sum_{i=1}^{n} \frac{\partial g_1}{\partial x_i} dx_i = 0 \to$

$g_2(x_1, \dots, x_n) = 0$ ... $\sum_{j=1}^{n} \frac{\partial g_2}{\partial x_j} dx_j = 0 \to$

} dx_1, dx_2 pomocí $dx_3, \dots, dx_n$

$x^0 \in D_f$ je lok. extrém $\Rightarrow \exists \lambda_1, \lambda_2 : \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

$F := f - \lambda_1 g_1 - \lambda_2 g_2$

$D^2(x_i x_j) F = a_{ij}$ ... $\sum a_{ij} dx_i dx_j \Rightarrow \sum \tilde{a}_{ij} dx_i dx_j$ < pos. def. / neg. def. / indef. / semidef.

① $f(x,y) = x^2 + y^2$ $\nabla f = (2x, 2y) = \vec{\emptyset}$ pro $(x,y) = \underline{(0,0)}$

$(Hf)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ... pos. def. → minimum

$f(x,y) = x^2 - y^2$ $\nabla f = (2x, -2y)$ ... $= \vec{\emptyset}$ v $\underline{(0,0)}$

$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ ... indefinitní → není extrém

$f(x,y) = -x^2 - y^2$ $\nabla f = (-2x, -2y)$ ... $= \vec{\emptyset}$ v $\underline{(0,0)}$

$Hf(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ ... neg. def. → maximum

② $f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$ → $x = y$: $4x^3 - 4x = 4x(x^2-1)$

$\nabla f = (4x^3 - 2x - 2y, 4y^3 - 2y - 2x)$

$\frac{\partial f}{\partial x} = 4x^3 - 2x - 2y = 0$

$\frac{\partial f}{\partial y} = 4y^3 - 2x - 2y = 0$

$4x^3 - 4y^3 = 0$

$\Rightarrow x = y$

$\nabla f = \vec{\emptyset} \Leftrightarrow$ $\left.\begin{aligned} x(2x^2-1) &= y \\ y(2y^2-1) &= x \end{aligned}\right\} \to$ $(0,0)$, $(1,1)$, $(-1,-1)$

$\frac{2x^2-1}{2y^2-1} = \frac{\frac{y}{x}}{\frac{x}{y}} = \frac{y^2}{x^2}$ ... $x^2(2x^2-1) = y^2(2y^2-1)$

$Hf = \begin{pmatrix} 12x^2-2 & -2 \\ -2 & 12y^2-2 \end{pmatrix} = 2\begin{pmatrix} 6x^2-1 & -1 \\ -1 & 6y^2-1 \end{pmatrix}$

$Hf(0,0) = 2\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$ ... $\det = 0$ ⇒ ~~není extrém~~ negativně semidefinitní (!) (nevíme)(obecně)

$Hf(1,1) = 2\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ ... $\det > 0$ pos. def. ⇒ minimum (ostré)

$Hf(-1,-1) = 2\begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}$ ... $\det > 0$ pos. def. ⇒ minimum (ostré)

• v $(0,0)$ $x^4 + y^4 - (x+y)^2$: $x = -y = \varepsilon$: $f > 0$; $x = y = \varepsilon$: $f < 0$ } není extrém v $(0,0)$

Kopáček uč. 2 pp. 132–133

Symetrická kvadratická forma $\varphi(h) = a h_1^2 + 2b h_1 h_2 + c h_2^2$ $a, b, c \in \mathbb{R}$

matice $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ je (1) definitní $\Leftrightarrow b^2 - ac = -\det A < 0$

- pozitivní pro $a, c > 0$
- negativní pro $a, c < 0$

(2) indefinitní $\Leftrightarrow b^2 - ac = -\det A > 0$

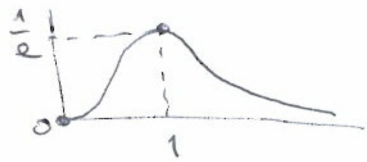
(3) semidefinitní, ale ne definitní $\Leftrightarrow -\det A = 0$

③ $(x^2+y^2)e^{-(x^2+y^2)} \to r^2 e^{-r^2}$ ... osově symetrická

$(r^2 e^{-r^2})' = 2r e^{-r^2} - 2r^3 e^{-r^2} = 2r(1-r^2)e^{-r^2}$

$\to 0$ pro $r=0$ a $r=1$

$\uparrow$ min $\quad \uparrow$ max



$\to$ • ostré minimum v $(0,0)$

• "kruhové" maximum pro $x^2+y^2=1$

④ $\left(2x^2 - xy + \frac{y^3}{3} - 5x + \frac{5y}{3} + \frac{10}{3}\right)e^{x+y} \longrightarrow$ $\left(\frac{1}{2}, -\frac{5}{2}\right)$ saddle $f = \frac{1}{2e^2}$

$(1,-1)$ minimum $f = 0$

$$\frac{\partial f}{\partial x} = \left(4x - y - 5 + 2x^2 - xy + \frac{y^3}{3} - 5x + \frac{5y}{3} + \frac{10}{3}\right)e^{x+y}$$

$$\frac{\partial f}{\partial y} = \left(-x + y^2 + \frac{5}{3} + 2x^2 - xy + \frac{y^3}{3} - 5x + \frac{5y}{3} + \frac{10}{3}\right)e^{x+y}$$

$$\to \begin{cases} 2x^2 - xy + \frac{y^3}{3} - x + \frac{2}{3}y - \frac{5}{3} = 0 \\ 2x^2 - xy + \frac{y^3}{3} - 6x + y^2 + \frac{5}{3}y + \frac{15}{3} = 0 \end{cases}$$

$\frac{1095}{1800} e^{47/60}$

$-x + \frac{2}{3}y - \frac{5}{3} = -6x + y^2 + \frac{5}{3}y + \frac{15}{3}$

$5x - y - \frac{20}{3} = y^2 \quad \longrightarrow \quad x = \frac{77}{60} \quad y = -\frac{1}{2}$

$5x = y^2 + y + \frac{20}{3}$

$x = \frac{1}{5}\left(y^2 + y + \frac{20}{3}\right)$

(5) $f(x,y) = \begin{cases} xy \ln(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$\frac{\partial f}{\partial x} = y\left(\frac{2x^2}{x^2+y^2} + \ln(x^2+y^2)\right)$ $\rightarrow \ln(x^2+y^2) + \frac{2x^2}{x^2+y^2} = 0$

$\frac{\partial f}{\partial y} = x\left(\frac{2y^2}{x^2+y^2} + \ln(x^2+y^2)\right)$ $\rightarrow \ln(x^2+y^2) + \frac{2y^2}{x^2+y^2} = 0$

$\frac{2(x^2-y^2)}{x^2+y^2} = 0$

$\nabla f = 0 \iff$
- $x = 0$ & $y = \pm 1$
- $y = 0$ & $x = \pm 1$
- $x^2 = y^2$ ... $1 + \ln(2x^2) = 0$ ... $x^2 = \frac{1}{2e} = y^2$

Podezřelé body:
- $(0, \pm 1)$ ... 2p
- $(\pm 1, 0)$ ... 2p
- $\left(\pm\frac{1}{\sqrt{2e}}, \pm\frac{1}{\sqrt{2e}}\right)$ ... 4p.

$\frac{\partial^2 f}{\partial y^2} = \frac{2xy(y^2+3x^2)}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x^2} = \frac{2xy(x^2+3y^2)}{(x^2+y^2)^2}$

$\frac{\partial^2 f}{\partial x \partial y} = \ln(x^2+y^2) + \frac{2(x^4+y^4)}{(x^2+y^2)^2}$

$$Hf = \begin{pmatrix} \frac{2xy(x^2+3y^2)}{(x^2+y^2)^2} & \ln(x^2+y^2) + \frac{2(x^4+y^4)}{(x^2+y^2)^2} \\ \ln(x^2+y^2) + \frac{2(x^4+y^4)}{(x^2+y^2)^2} & \frac{2xy(y^2+3x^2)}{(x^2+y^2)^2} \end{pmatrix}$$

$\left.\begin{matrix} Hf(0,\pm 1) \\ Hf(\pm 1, 0) \end{matrix}\right\} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ ... det < 0 ... indef. ⇒ sedlový bod, $f(x,y) = 0$

$\left.\begin{matrix} Hf\left(+\frac{1}{\sqrt{2e}}, +\frac{1}{\sqrt{2e}}\right) \\ Hf\left(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right) \end{matrix}\right\} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ ... det > 0 ... def. poz. ⇒ minimum, $f(x,y) = -\frac{1}{2e}$

$\left.\begin{matrix} Hf\left(+\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}}\right) \\ Hf\left(-\frac{1}{\sqrt{2e}}, +\frac{1}{\sqrt{2e}}\right) \end{matrix}\right\} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ ... det > 0 ... neg. def. ⇒ maximum, $f(x,y) = \frac{1}{2e}$

Speciálně, pro $(x,y) = (0,0)$ ... $f(x,y) = 0$

- $x = y$: $x^2 \ln(2x^2)$ $\lim_{x\to 0} \frac{f(x,x) - f(0,0)}{x} = \lim_{x \to 0} x \ln(2x^2) = 0$
- $x = -y$: $-x^2 \ln(2x^2)$ $\lim_{x\to 0} \frac{f(x,-x) - f(0,0)}{x} = \lim_{x \to 0} -x \ln(2x^2) = 0$

$\tilde{f}' = 2x\ln(2x^2) + \frac{2x^3 \cdot 2}{2x^2} = 2x(\ln(2x^2)+1)$ $\tilde{f}'' = 2(\ln(2x^2)+3)$

$\tilde{\tilde{f}}' = -2x(\ln(2x^2)+1)$ $\tilde{\tilde{f}}'' = -2(\ln(2x^2)+3)$

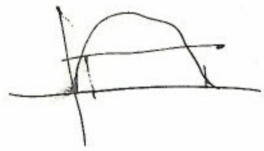
$f(x,y) = xy \ln(x^2+y^2)$, $\ln(x^2+y^2) < 0$

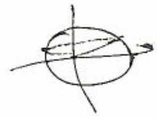
Není extrém v (0,0)

$f(0,0) = 0$ xy > 0 I, III kv.; < 0 II, IV kv.

$f(x,y)$ < 0, > 0

⑥ $f(x,y) = x + y + 4\cos x \cos y$





$$\frac{\partial f}{\partial x} = 1 - 4\sin x \cos y \stackrel{!}{=} 0$$

$$\frac{\partial f}{\partial y} = 1 - 4\cos x \sin y \stackrel{!}{=} 0$$

$\ominus$ $\quad \sin x \cos y - \cos x \sin y = 0 \rightarrow \sin(x-y) = 0 \rightarrow x - y = k\pi$

$\oplus$ $\quad 2 - 4\sin(x+y) = 0 \rightarrow \sin(x+y) = \frac{1}{2} \rightarrow \begin{cases} x+y = \frac{\pi}{6} + 2l\pi \\ x+y = \frac{5}{6}\pi + 2l\pi \end{cases}$

$x = y + k\pi$

$$\rightarrow \quad 2y + k\pi = \frac{\pi}{6} + 2l\pi \rightarrow 2y = \frac{\pi}{6} + (2l-k)\pi$$

$$2y + k\pi = \frac{5}{6}\pi + 2l\pi \rightarrow 2y = \frac{5\pi}{6} + (2l-k)\pi$$

$$y = \frac{\pi}{12} + \left(l - \frac{k}{2}\right)\pi$$

$$y = \frac{5\pi}{12} + \left(l - \frac{k}{2}\right)\pi$$

} všech kombinace ...

k = 0

$y = \frac{\pi}{12} + l\pi$

$y = \frac{5\pi}{12} + l\pi$

y	x	
$\frac{\pi}{12} + 2l\pi$	$= y$	1
$-\frac{11\pi}{12} + 2l\pi$	$= y$	2
$\frac{5\pi}{12} + 2l\pi$	$= y$	3
$-\frac{7\pi}{12} + 2l\pi$	$= y$	4

k = −1 ...

⑦ $f(x,y) = $ ~~[struck out]~~ $\sin x + \cos y + \cos(x-y)$ $\qquad (0, \frac{\pi}{2}) \times (0, \frac{\pi}{2})$

$$\frac{\partial f}{\partial x} = \cos x - \sin(x-y) = 0$$

$$\frac{\partial f}{\partial y} = -\sin y + \sin(x-y) = 0$$

$$\cos x = \sin y \quad \to \quad x = \frac{\pi}{2} - y$$

$\cos\frac{\pi}{2}\sin 2y$

$$\to \cos\left(\frac{\pi}{2} - y\right) - \sin\left(\frac{\pi}{2} - 2y\right) = \underbrace{\cos\frac{\pi}{2}}_{0}\cos y + \underbrace{\sin\frac{\pi}{2}}_{1}\sin y - \underbrace{\sin\frac{\pi}{2}}_{1}\cos 2y + \text{[struck out]}$$

$$\sin y - \cos 2y = 0 \qquad \cos 2y = 1 - 2\sin^2 y$$

$$\sin y - 1 + 2\sin^2 y = 0$$

$$\frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{2} = \begin{cases} \frac{1}{2} = \sin y \;\ldots\; y = \frac{\pi}{6} \;\ldots\; x = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \\ -1 = \sin y \;\ldots\; y = \pi \;\notin (0, \pi/2) \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x - \cos(x-y)$$

Podezřelý bod: $\left(\frac{\pi}{3}, \frac{\pi}{6}\right)$

$$\frac{\partial^2 f}{\partial y^2} = -\cos x - \cos(x-y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = +\cos(x-y)$$

$$Hf = \text{[struck out]} \begin{pmatrix} -\sin x - \cos(x-y) & \cos(x-y) \\ \cos(x-y) & -\cos x - \cos(x-y) \end{pmatrix}$$

$$Hf\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \begin{pmatrix} -\sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}(1-\sqrt{3}) \end{pmatrix}$$

$$\det H\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \sqrt{3}\,\frac{1}{2}(1-\sqrt{3}) - \frac{3}{4} < 0$$

(8) $x - 2y + \ln\sqrt{x^2+y^2} + 3\operatorname{arctan}\frac{y}{x}$, $x \neq 0$

$$\frac{\partial f}{\partial x} = 1 + \frac{1}{\sqrt{x^2+y^2}}\,\frac{2x\,\frac{1}{2}}{\sqrt{x^2+y^2}} + 3\,\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= 1 + \frac{x}{x^2+y^2} + \frac{3x^2(-y)}{x^2(x^2+y^2)} = \frac{x^2+y^2+x-3y}{x^2+y^2} \overset{!}{=} 0$$

$$\frac{\partial f}{\partial y} = -2 + \frac{y}{x^2+y^2} + 3\,\frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x} = \frac{-2x^2-2y^2+y+3x}{x^2+y^2} \overset{!}{=} 0$$

$\leadsto$ $x^2+y^2+x-3y = 0 \quad |\cdot 2$

$-2x^2-2y^2+y+3x = 0$

$5x - 5y = 0$

$\underline{x = y} \neq 0$ $\leadsto$ $2x^2 - 2x = 0$

$2x(x-1) = 0$

$x = 0 \notin D_f$ $\qquad$ $x = 1 = y$

$\underline{(1,1)}$ ... stacionární bod

$$\frac{\partial^2 f}{\partial x^2} = \dots = \frac{-x^2+6xy+y^2}{(x^2+y^2)^2}\bigg|_{(1,1)} = \frac{3}{2}$$

$$\frac{\partial^2 f}{\partial y^2} = \dots = \frac{x^2-6xy-y^2}{(x^2+y^2)^2}\bigg|_{(1,1)} = -\frac{3}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \dots = \frac{-3x^2-2xy+3y^2}{(x^2+y^2)^2}\bigg|_{(1,1)} = -\frac{1}{2}$$

$$Hf(1,1) = \underbrace{\frac{1}{(x^2+y^2)^2}}_{\frac{1}{4}}\begin{pmatrix} 6 & -2 \\ -2 & -6 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$|Hf(1,1)| = \frac{1}{4}\cdot(-36-4) = -10 \Rightarrow$ indefinitní

$\Rightarrow$ sedlový bod

⑨ $f(x,y,z) = x^2+y^2+z^2+2x+4y-6z$

$$\frac{\partial f}{\partial x} = 2x+2 = 0 \quad \ldots \; x=-1$$
$$\frac{\partial f}{\partial y} = 2y+4 = 0 \quad \ldots \; y=-2$$
$$\frac{\partial f}{\partial z} = 2z-6 = 0 \quad \ldots \; z=3$$

$$Hf(x,y,z) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \ldots \text{pos. def.} \Rightarrow \text{lok. minimum}$$

⑩ $(ax+by+cz)e^{-x^2-y^2-z^2} \quad \in C^\infty(\mathbb{R}^3)$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2-z^2}\left(a-2x(ax+by+cz)\right) \stackrel{!}{=} 0 \quad /\cdot x$$
$$\frac{\partial f}{\partial y} = e^{-x^2-y^2-z^2}\left(b-2y(ax+by+cz)\right) \stackrel{!}{=} 0 \quad /\cdot y$$
$$\frac{\partial f}{\partial z} = \underbrace{e^{-x^2-y^2-z^2}}_{\text{always } >0}\underbrace{\left(c-2z(ax+by+cz)\right)}\stackrel{!}{=} 0 \quad /\cdot z$$

$$ax+by+cz = 2(x^2+y^2+z^2)(ax+bx+cz)$$

$$\left\langle \begin{array}{ll} ax+by+cz=0 & (x,y,z)=(0,0,0) \\ x^2+y^2+z^2=\frac{1}{2} & \ldots (x,y,z)\neq(0,0,0) \end{array}\right.$$

}

Implicitní fce

Věta (o ∃ spojité, 12.4.2)

$F: \mathbb{R}^{N+1} \to \mathbb{R}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}$

Nechť $F(a,b) = 0$ a ∃ okolí (a,b), kde F spojité a $y \to F(x,y)$ ryze monotónní.

Pak ∃ $\delta, \Delta > 0$ takové, že $\forall x \in U_\delta(a)$ ∃! $y_x \in U_\Delta(b)$ splňující $F(x,y_x) = 0$.

Navíc $x \to y_x$ je spojitá na $U_\delta(a)$.

Věta (o derivaci fce, 12.4.6)

$F: \mathbb{R}^{N+1} \to \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}$. ← podle poslední proměnné

Nechť $F(a,b) = 0$, ∃ $U(a,b)$, kde F je třídy C^k a $\frac{\partial F}{\partial y}(a,b) \neq 0$.

Pak ∃ $\delta, \Delta > 0$ takové, že $\forall x \in U_\delta(a)$ ∃! $y_x \in U_\Delta(b)$ splňující $F(x,y_x) = 0$

a funkce $\varphi: x \to y_x$ je třídy C^k na $U_\delta(a)$.

Navíc
$$\frac{\partial \varphi}{\partial x_j}(x) = \frac{\frac{\partial F}{\partial x_j}(x,\varphi(x))}{\frac{\partial F}{\partial y}(x,\varphi(x))} \quad \forall j = \{1,\dots,N\} \text{ a } x \in U_\delta(a).$$

Věta (o implicitní fci) (12.4.13)

Nechť $N, m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, $F: \mathbb{R}^{N+m} \to \mathbb{R}^m$, $a \in \mathbb{R}^N$, $b \in \mathbb{R}^m$.

Nechť $F(a,b) = (0,\dots,0)$, ∃ okolí (a,b), kde všechny složky zobrazení F jsou C^k,

a
$$\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1}(a,b) & \frac{\partial F_1}{\partial y_2}(a,b) & \cdots & \frac{\partial F_1}{\partial y_m}(a,b) \\ \frac{\partial F_2}{\partial y_2}(a,b) & \frac{\partial F_2}{\partial y_2}(a,b) & \cdots & \frac{\partial F_2}{\partial y_m}(a,b) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1}(a,b) & \frac{\partial F_m}{\partial y_2}(a,b) & \cdots & \frac{\partial F_m}{\partial y_m}(a,b) \end{pmatrix} \neq 0$$

Pak ∃ $\delta, \Delta > 0$ tak, že $\forall x \in U_\delta(a)$ ∃! $y_x \in U_\Delta(b)$ splňující $F(x,y_x) = (0,\dots,0)$

a pro zobrazení $\varphi: x \to y_x$ platí, že $\varphi \in C^k(U_\delta(a); \mathbb{R}^m)$

(11) Dokažte, že existuje okolí V bodu (1,1) takové, že množina
$\{(x,y) : x^3+y^3-2xy = 0\} \cap V$ je grafem nějaké funkce, která je třídy
C^2 na nějakém okolí bodu 1. Spočtěte $f'(1)$, $f''(1)$.

- $g(x,y) = x^3+y^3-2xy$
- $g(1,1) = 0$
- $\frac{\partial g(x,y)}{\partial y} = 3y^2-2x \rightsquigarrow \frac{\partial g}{\partial y}(1,1) = 1 \neq 0 \quad \& \quad g(x,y) \in C^\infty(\mathbb{R}^2)$

$\Rightarrow \exists\, y = f(x),\ f(x) \in C^\infty$ na okolí $x=1$

- $x^3 + f^3(x) - 2x f(x) = 0 \quad \Big| \frac{\partial}{\partial x}$

$$3x^2 + 3f^2(x)\frac{\partial f}{\partial x} - 2f(x) - 2x\frac{\partial f}{\partial x} = 0$$

$$\rightsquigarrow \frac{\partial f}{\partial x} = \frac{2f(x)-3x^2}{3f^2(x)-2x} \qquad \frac{\partial f}{\partial x}(1) = \frac{2-3}{3-2} = \underline{\underline{-1}}$$

- $$\frac{\partial^2 f}{\partial x^2} = \frac{\left(2\frac{\partial f}{\partial x} - 6x\right)\left(3f^2(x)-2x\right) - \left(2f(x)-3x^2\right)\left(6f(x)\frac{\partial f}{\partial x} - 2\right)}{\left(3f^2(x)-2x\right)^2}$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = \frac{(-2-6)(3-2)-(2-3)(-6-2)}{(3-2)^2} = \frac{-8-(8)}{1} = \underline{\underline{-16}}$$

Or we could do: $x = \varphi(y)$

- $\frac{\partial g(x,y)}{\partial x} = 3x^2 - 2y \qquad \frac{\partial g}{\partial x}(1,1) = 3-2 = 1 \neq 0$

$\Rightarrow \exists\, x = \varphi(y)$

$$\varphi^3(y) + y^3 - 2\varphi(y)y = 0 \quad \Big| \frac{\partial}{\partial y}$$

$$3\varphi^2(y)\frac{\partial \varphi(y)}{\partial y} + 3y^2 - 2\varphi(y) - 2y\frac{\partial \varphi(y)}{\partial y} = 0$$

$\vdots$

(12) Dokažte, že existuje okolí V bodu $(3,-2,2)$ takové, že množina $\{(x,y,z) : z^3 - xz + y = 0\} \cap V$ je grafem nějaké funkce, která je C^2 na okolí bodu $(3,-2)$. Spočtěte $\frac{\partial^2 z}{\partial y^2}(3,-2)$

- $g(x,y,z) = z^3 - xz + y$ a $g(3,-2,2) = 8 - 6 - 2 = 0$ ✓
- $\frac{\partial g}{\partial z} = 3z^2 - x \rightarrow \frac{\partial g}{\partial z}(3,-2,2) = 3 \cdot 2^2 - 3 = 12 - 3 = 9 \neq 0$

$\&\ g \in C^\infty(\mathbb{R}^3) \Rightarrow \exists z(x,y)$

- $\frac{\partial g}{\partial y} = 0 = 3z^2 \frac{\partial z}{\partial y} - x \frac{\partial z}{\partial y} + 1 \rightarrow \frac{\partial z}{\partial y} = \frac{1}{x - 3z^2}$

$\frac{\partial z}{\partial y}(3,-2,2) = \frac{1}{3 - 3 \cdot 2^2} = -\frac{1}{9}$

- $\frac{\partial^2 z}{\partial y^2} = \frac{-\left(-6z \frac{\partial z}{\partial y}\right)}{(x-3z^2)^2} = \frac{6z \frac{\partial z}{\partial y}}{(x-3z^2)^2}$

$\frac{\partial^2 z}{\partial y^2}(3,-2,2) = \frac{6 \cdot 2 \cdot \left(-\frac{1}{9}\right)}{(3 - 3 \cdot 2^2)^2} = -\frac{12 \cdot \frac{1}{9}}{(-9)^2} = -\frac{4}{3}\frac{1}{81} = -\frac{4}{243}$

(13) Spočtěte parciální derivace 2. řádu funkce implicitně zadané vztahem:

$x + y + z = e^{-(x+y+z)}$

- $g(x,y,z) = x + y + z - e^{-(x+y+z)}$
- $\frac{\partial g}{\partial z} = \underbrace{1 + e^{-(x+y+z)}}_{\neq 0\ (>0\ \forall x,y,z)}$ $\&\ g(x,y,z) \in C^\infty$

- $\frac{\partial g(x,y,z(x,y))}{\partial x} = 1 + \frac{\partial z}{\partial x} + e^{-(x+y+z)}\left(1 + \frac{\partial z}{\partial x}\right) = 0$

$\rightarrow \left(1 + \frac{\partial z}{\partial x}\right)\underbrace{\left(1 + e^{-(x+y+z)}\right)}_{>0} = 0$

$\Rightarrow \frac{\partial z}{\partial x} = -1$

$\frac{\partial z}{\partial y} = -1$ ... ze symetrie

$\frac{\partial x}{\partial y} = -1$ ... sym

$\frac{\partial x}{\partial y} = \frac{\partial y}{\partial z} = \frac{\partial z}{\partial x} = -1$

(14) Nalezněte 1. a 2. diferenciál funkce dané vztahem $z = x + \operatorname{arctan} \frac{y}{z-x}$

$z - x = \operatorname{arctan} \frac{x}{z-x}$

$(z, \frac{\pi}{4}, 1) = (x_0, y_0, z_0)$

- $g(x,y,z) = z - x - \operatorname{arctan} \frac{y}{z-x}$ ... $x \neq z$
- $\frac{\partial g}{\partial z} = 1 - \frac{1}{1+\left(\frac{y}{z-x}\right)^2} \cdot \frac{y}{-(z-x)^2} = 1 + \frac{y}{(z-x)^2 + y^2} \neq 0$ pro $\frac{y}{(z-x)^2 - y^2} \neq -1$

$\rightsquigarrow z = z(x,y)$

- $\frac{\partial g(x,y,z(x,y))}{\partial x} = \frac{\partial z}{\partial x} - 1 + \frac{y}{(z-x)^2+y^2}\left(\frac{\partial z}{\partial x} - 1\right) = \left(\frac{\partial z}{\partial x} - 1\right)\underbrace{\left(1 + \frac{y}{(z-x)^2+y^2}\right)}_{\neq 0 \text{ (podm. pro impl. fce)}} = 0$

$\Rightarrow \boxed{\frac{\partial z}{\partial x} = 1}$

- $\frac{\partial g(x,y,z(x,y))}{\partial y} = \frac{\partial z}{\partial y} - \frac{1}{1+\left(\frac{y}{z-x}\right)^2}\left(\frac{1}{z-x} - \frac{y}{(z-x)^2}\frac{\partial z}{\partial y}\right)$

$= \frac{\partial z}{\partial y} - \frac{(z-x)^2}{(z-x)^2+y^2}\left(\frac{1}{z-x} - \frac{y}{(z-x)^2}\frac{\partial z}{\partial y}\right)$

$= \frac{\partial z}{\partial y}\left(1 + \frac{y}{(z-x)^2+y^2}\right) - \frac{z-x}{(z-x)^2+y^2} = 0$

$\rightsquigarrow \boxed{\frac{\partial z}{\partial y} = \frac{z-x}{(z-x)^2+y^2+y}}$

$\frac{\partial z}{\partial x} = 1 \Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2} = 0} \quad \boxed{\frac{\partial^2 z}{\partial x \partial y} = 0}$

$V = (z-x)^2 + y^2$

- $\frac{\partial^2 z}{\partial y^2} = \ldots = \frac{\frac{\partial z}{\partial y}\left((z-x)^2+y^2+y\right) - 2(z-x)^2\frac{\partial z}{\partial y} - (2y+1)(z-x)}{\left((z-x)^2+y^2+y\right)^2}$

$= \frac{z-x - 2\frac{(z-x)^3}{V} - (2y+1)(z-x)}{V^2}$ $\quad \frac{\partial z}{\partial y} = \ldots$

$= \frac{z-x}{V^3}\left(V - 2(z-x)^2 - (2y+1)V\right)$

$= \frac{z-x}{V^3}\left(-2(z-x)^2 - 2y\left((z-x)^2+y^2\right)\right) = \frac{\partial^2 z}{\partial x^2}$

$\Rightarrow d^2 z = \frac{\partial^2 z}{\partial y^2}(dy)^2$

Kvadratická forma 2. diferenciálu: $d^2 z(a)(h) = z_{xx}(a) h_1^2 + 2 z_{xy}(a) h_1 h_2 + z_{yy}(a) h_2^2$

(15) Jsou-li $x = f(y,z)$, $y = g(x,z)$, $z = h(x,y)$ implicitně zadány vztahem $F(x,y,z) = 0$, ukažte, že $f_y g_z h_x = -1$.

- $F(f(y,z),y,z)$: $\frac{\partial F}{\partial y} = \frac{\partial F}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial F}{\partial y} = 0 \Rightarrow f_y = -\frac{F_y}{F_x}$

- $F(x,g(x,z),z)$: $\frac{\partial F}{\partial z} = \frac{\partial F}{\partial y}\frac{\partial g}{\partial z} + \frac{\partial F}{\partial z} = 0 \Rightarrow g_z = -\frac{F_z}{F_y}$

- $F(x,y,h(x,y))$: $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial z}\frac{\partial h}{\partial x} + \frac{\partial F}{\partial x} = 0 \Rightarrow h_x = -\frac{F_x}{F_z}$

$$f_y g_z h_x = \left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)\left(-\frac{F_x}{F_z}\right) = \underline{-1}$$

(16) Napište du a dv, je-li $u + v = x + y$, $\frac{\sin u}{\sin v} = \frac{x}{y}$

$g(x,y,u,v): \mathbb{R}^{2+2} \to \mathbb{R}^2$ $g_1(x,y,u,v) = u + v - x - y$

$g_2(x,y,u,v) = \frac{\sin u}{\sin v} - \frac{x}{y}$ $y \neq 0$, $\sin v \neq 0$

$$\begin{vmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{\cos u}{\sin v} & -\frac{\sin u \cos v}{\sin^2 v} \end{vmatrix} = -\frac{\sin u \cos v}{\sin^2 v} - \frac{\cos u}{\sin v}$$

$$= \frac{-1}{\sin^2 v}\left(\sin u \cos v - \cos u \sin v\right) = -\frac{1}{\sin^2 v}\sin(u-v) \overset{!}{\neq} 0$$

$u - v \neq k\pi$

- $u(x,y) \to du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

  $v(x,y) \to dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

- $\frac{\partial g_1(x,y,u(x,y),v(x,y))}{\partial x} = 0 = u_x + v_x - 1$ $\frac{\partial g_2}{\partial x} = \frac{\cos u}{\sin v}u_x - \frac{\sin u \cos v}{\sin^2 v}v_x - \frac{1}{y} = 0$

  $\frac{\partial g_1}{\partial y} = 0 = u_y + v_y - 1$ $\frac{\partial g_2}{\partial y} = \frac{\cos u}{\sin v}u_y - \frac{\sin u \cos v}{\sin^2 v}v_y + \frac{x}{y^2} = 0$

$\to$ 4 equations for u_x, u_y, v_x, v_y

• $u_x + v_x = 1$

$$\frac{\cos u}{\sin v} u_x - \frac{\sin u \cos v}{\sin^2 v} v_x = \frac{1}{y}$$

$$\left(\frac{\sin u \cos v}{\sin^2 v} + \frac{\cos u}{\sin v} \right) u_x = \frac{1}{y} + \frac{\sin u \cos v}{\sin^2 v}$$

$$(\sin u \cos v + \cos u \sin v) u_x = \frac{\sin^2 v}{y} + \sin u \cos v$$

$$u_x = \frac{\frac{\sin^2 v}{y} + \sin u \cos v}{\sin(u+v)}$$

$$v_x = 1 - u_x$$

• $u_y + v_y = 1$

$$\frac{\cos u}{\sin v} u_y - \frac{\sin u \cos v}{\sin^2 v} = -\frac{x}{y^2}$$

$$\sin(u+v) u_y = -\frac{x}{y^2} \sin^2 v + \sin u \cos v$$

$$u_y = \frac{\sin u \cos v - \frac{x}{y^2} \sin^2 v}{\sin(u+v)}$$

$$v_y = 1 - u_y$$

$\rightarrow$ $du = u_x \, dx + u_y \, dy$

$dv = (1-u_x) dx + (1-u_y) dy$

(17) Hledejte lokální extrémy funkce $z = z(x,y)$, dané implicitně vztahem $\underbrace{(x^2+y^2+z^2)}_{r^2}{}^2 = a^2(x^2+y^2-z^2)$

$g(x,y,z) = (x^2+y^2+z^2)^2 - a^2(x^2-y^2-z^2)$

$\frac{\partial g}{\partial z} = 4r^2 z + 2a^2 z = 2z(2r^2+a^2) \neq 0$ pro $\underline{z \neq 0}$

→ lok. extrémy: need z_x, z_y

$\frac{\partial g(x,y,z(x,y))}{\partial x} = 2r^2(2x+2zz_x) - a^2(2x-2zz_x) = 0$

$\rightarrow \underline{z_x = \frac{2r^2-a^2}{2r^2+a^2}\frac{x}{z}} \quad \dots \quad \underline{z_y = \frac{2r^2-a^2}{2r^2+a^2}\frac{y}{z}}$

$z_x \overset{!}{=} 0 \;\&\; z_y \overset{!}{=} 0 \quad \rightarrow \quad \underline{2r^2 = a^2 \quad \dots \quad r^4 = \frac{a^4}{4}}$

Vztah: $r^4 = a^2(x^2+y^2-z^2) = a^2(-x^2-y^2-z^2+2x^2+2y^2) = a^2(-r^2+2x^2+2y^2)$