Funkce více proměnných

Totální diferenciál

V následujících příkladech zjistěte, kde má funkce totální diferenciál. Určete ho

1. $f(x, y) = \ln(x + y)$
2. $f(x, y, z) = \cos x \cosh y$
3. $f(x, y) = |x||y|$
4. $f(x, y) = \sqrt[3]{xy}$
5. $f(x, y) = \sqrt[5]{x^5 + y^5}$
6. $f(x, y, x) = x^{\frac{y}{z}}$.
7. Nechť $\alpha \in \mathbb{R}$. Pro jaké hodnoty α bude mít funkce
$$f(x, y) = (x^2 + y^2)^\alpha \sin \frac{1}{x^2 + y^2}$$
totální diferenciál 1. řádu v bodě $(0, 0)$?
8. Napište diferenciál funkce $f(x, y, z)$, kde $x = u^2 + v^2$, $y = u^2 - v^2$, $z = 2uv$.
9. Nechť f má totální diferenciál v bodě (1,1) a $g(t, u) = f(f(u, t), f(t, u))$. Vypočtěte $\frac{\partial g}{\partial x_1}(1, 1)$, je-li $f(1, 1) = \frac{\partial f}{\partial x_1}(1, 1) = 1$, $\frac{\partial f}{\partial x_2}(1, 1) = 2$.
10. Spočtěte $d^3 f$, je-li $f(x, y, z) = xyz$.
11. Pomocí diferenciálu spočtěte přibližně
(a) $1,02^2.2,003^3.3,004^3$ (b) $\sin 29° \cdot \mathrm{tg}\, 46°$

Obyčejné diferenciální rovnice

Rovnice ve tvaru totálního diferenciálu

Nalezněte obecná řešení rovnic. Pokud nejsou ve tvaru totálního diferenciálu, hledejte vhodný integrační faktor

12.
$$2xy\,\mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y = 0$$

13.
$$\mathrm{e}^{-y}\,\mathrm{d}x - (2y + x\mathrm{e}^{-y})\,\mathrm{d}y = 0$$

14.
$$\frac{3x^2 + y^2}{y^2}\,\mathrm{d}x - \frac{2x^3 + 5y}{y^3}\,\mathrm{d}y = 0$$

15.
$$(x^2 + y)\,\mathrm{d}x - x\,\mathrm{d}y = 0, \quad \mu = \mu(x)$$

16.
$$(xy^2 + y)\,\mathrm{d}x - x\,\mathrm{d}y = 0, \quad \mu = \mu(y)$$

17.
$$(x^2+x^2y+2xy-y^2-y^3)\,\mathrm{d}x+(y^2+xy^2+2xy-x^2-x^3)\,\mathrm{d}y = 0, \quad \mu = \mu(x+y)$$

18.
$$x^2y^3 + y + (x^3y^2 - x)y' = 0, \quad \mu = \mu(xy).$$

Počítání 2/11 – Funkce více proměnných – Totální diferenciál

① – ⑦ ... Úlohy jsme vyřešili v sadě 2/10.

⑧ Najděte diferenciál funkce $f(x,y,z)$, kde
$x = u^2 + v^2$, $y = u^2 - v^2$, $z = 2uv$

- $f(x,y,z)$ ... $f: \mathbb{R}^3 \to \mathbb{R}$
- $\vec{g}(u,v) = (x,y,z)(u,v) = (x(u,v), y(u,v), z(u,v))$ ... $\vec{g}: \mathbb{R}^2 \to \mathbb{R}^3$
- $df(x,y,z)$ ... $df(x,y,z)h = \nabla f \cdot h$, $h = (h_1,h_2,h_3)$
- $F(u,v) = f(g) = f \circ g$

$\rightsquigarrow d(f\circ g)(a) = d f(g(a)) \circ dg(a)$

$df = \nabla f$ (3 vektor) $\qquad dg = \nabla g$ (3x2 matice)

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\nabla \vec{g} = \begin{pmatrix} \frac{\partial g_1}{\partial u} & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u} & \frac{\partial g_2}{\partial v} \\ \frac{\partial g_3}{\partial u} & \frac{\partial g_3}{\partial v} \end{pmatrix}$$

$\to dF(u,v)h = \nabla f \cdot \nabla g \cdot h$ (2 vektor)

$= \cancel{\nabla f \cdot (dx(u,v)\ldots}$

$= \nabla f \cdot (dx, dy, dz) \cdot h$, $dx(u,v)$

$= \nabla f \cdot (2u h_1 + 2v h_2,\ 2u h_1 - 2v h_2,\ 2v h_1 + 2u h_2)$

$= \left(\frac{\partial f}{\partial x} 2u + \frac{\partial f}{\partial y} 2u + \frac{\partial f}{\partial z} 2v\right) h_1 + \left(\frac{\partial f}{\partial x} 2v - \frac{\partial f}{\partial y} 2v + \frac{\partial f}{\partial z} 2u\right) h_2$

"du" $\qquad$ "dv"

(9) Nechť má f TD v bodě $(1,1)$ a $g(t,u) = f(f(u,t), f(t,u))$.
Vypočtěte $\frac{\partial g}{\partial x_1}(1,1)$, je-li $f(1,1) = \underbrace{\frac{\partial f}{\partial x_1}(1,1)}_{1}$, $\frac{\partial f}{\partial x_2}(1,1) = 2$

$$\frac{\partial g}{\partial x_1} = \frac{\partial f}{\partial x_1}\underbrace{(\underbrace{f(u,t)}_{1}, \underbrace{f(t,u)}_{1})}_{1} \underbrace{\frac{\partial f}{\partial x_2}(u,t)}_{2} + \frac{\partial f}{\partial x_2}\underbrace{(\underbrace{f(u,t)}_{1}, \underbrace{f(t,u)}_{1})}_{1} \underbrace{\frac{\partial f}{\partial x_1}(t,u)}_{1}$$

$(u,t) = (1,1)$
$f(1,1) = 1$

$$= \frac{\partial f}{\partial x_1}(1,1)\frac{\partial f}{\partial x_2}(1,1) + \frac{\partial f}{\partial x_2}(1,1)\frac{\partial f}{\partial x_1}(1,1) = 2\frac{\partial f}{\partial x_1}(1,1)\frac{\partial f}{\partial x_2}(1,1) = 2\cdot 1\cdot 2 = \underline{4}$$

(10) Spočtěte $d^3 f$, je-li $f(x,y,z) = xyz$.

TD řádu 3

Obecní TD řádu k: $d^k f(a)(h^1, \dots, h^k) = \sum_{i_1,\dots,i_k=1}^{N} \frac{\partial^k f}{\partial x_{i_1}\dots\partial x_{i_k}}(a)\, h^1_{i_1}\dots h^k_{i_k}$ $\quad a \in \mathbb{R}^N$

$$\to d^3 f(x,y,z)(h^1,h^2,h^3) = \frac{\partial^3 f}{\partial x^3} h^1_x h^1_x h^2_x + \frac{\partial^3 f}{\partial x^2\partial y} h^1_y h^2_x h^3_x + \frac{\partial^3 f}{\partial x^2 \partial z} h^1_z h^2_x h^3_x$$
$$+ \frac{\partial^3 f}{\partial x^2 \partial y} h^1_x h^2_y h^3_x + \frac{\partial^3 f}{\partial x \partial y^2} h^1_y h^2_y h^3_x + \frac{\partial^3 f}{\partial x \partial y \partial z} h^1_z h^2_y h^3_x + \dots$$

Celkem 27 členů
$\underbrace{(3 \times 3 \times 3)}$
TD $\underline{3.}$ řádu fce $\underline{3}$ prom.

Avšak $f(x,y,z) = xyz$ a ze všech parc. der. 3. řádu je pouze jedna nenulová:
$\frac{\partial^3 f}{\partial x \partial y \partial z} = 1$ a ta odpovídá 6 členům (z oněch 27)

$$\to d^3 f(x,y,z)(h^1,h^2,h^3) = \underbrace{\frac{\partial^3 f}{\partial x \partial y \partial z}}_{=1}\left(h^1_x h^2_y h^3_z + h^1_y h^2_z h^3_x + h^1_z h^2_x h^3_y + h^1_x h^2_z h^3_y + h^1_y h^2_x h^3_z + h^1_z h^2_y h^3_x\right)$$

(11) Pomocí diferenciálu spočtěte přibližně:

(a) $1.02^2 \cdot 2.003^3 \cdot 3.004^3$ ... $f(x,y,z) = x^2 y^3 z^3$

$(x_0,y_0,z_0) = (1,2,3)$

$\to$ Taylor $f(x,y,z)$ v (x_0,y_0,z_0)

$$f(a+h) = f(a) + \sum_{|\alpha|=1} \binom{1}{\alpha} D^\alpha f(a) h^\alpha + \frac{1}{2!}\sum_{|\alpha|=2} \binom{2}{\alpha} D^\alpha f(a) h^\alpha + \dots + \frac{1}{m!} \sum_{|\alpha|=m} \binom{m}{\alpha} D^\alpha f(a) h^\alpha$$

$$+ \frac{1}{(m+1)!} \sum_{|\alpha|=m+1} \binom{m+1}{\alpha} D^\alpha f(a+\theta h) h^\alpha$$

$\alpha \in (0,1)$

$$= f(a) + \sum_{i=1}^{k} \frac{1}{i!} d^i f(a)(\underbrace{h,\dots,h}_{i\text{ krát}}) + \frac{1}{(k+1)!} d^{k+1} f(a+\theta h)(\underbrace{h,\dots,h}_{k+1})$$

$$\to f(1+\underset{h_x}{0.02}, 2+\underset{h_y}{0.003}, 3+\underset{h_z}{0.004}) =$$

$$= \underbrace{f(1,2,3) + \frac{\partial f}{\partial x}(1,2,3) h_x + \frac{\partial f}{\partial y}(1,2,3) h_y + \frac{\partial f}{\partial z}(1,2,3) h_z}_{} + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(1,2,3) h_x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,2,3) h_x h_y + \dots\right) + \dots$$

$$= 216 + 432 \times 0.02 + 324 \times 0.003 + 216 \times 0.004$$

$$= 216 + 8.64 + 0.972 + 0.864 = 226.476 \quad (\text{Exact: } 226.6433\dots)$$

(b) $\sin(29°) \cdot \tan(46°)$ ... $f(x,y) = \sin x \tan y$

$(x_0,y_0) = \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

$(h_x,h_y) = \left(-\frac{\pi}{180}, +\frac{\pi}{180}\right)$

$$f(x_0+h_x, y_0+h_x) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0) h_x + \frac{\partial f}{\partial y}(x_0,y_0) h_y + \dots$$

$$= \sin(x_0)\tan(y_0) + \cos(x_0)\tan(y_0) h_x + \frac{\sin(x_0)}{\cos^2(y_0)} h_y + \dots$$

$$= \frac{1}{2}\cdot 1 + \frac{\sqrt{3}}{2}\cdot 1 \left(-\frac{\pi}{180}\right) + \underbrace{\frac{1}{2}\left(\frac{2}{\sqrt{2}}\right)^2}_{1}\left(\frac{\pi}{180}\right)$$

$$= \frac{1}{2} + \left(1 - \frac{\sqrt{3}}{2}\right)\frac{\pi}{180} = 0.5 + 0.002338 = 0.50234$$

$(\text{Exact: } 0.50203\dots)$

Příklad (12)-(18): Rovnice ve tvaru totálního diferenciálu

Def. Rovnici $M(x,y)\,dx + N(x,y)\,dy = 0$ nazveme rovnicí ve tvaru totálního diferenciálu na otevřené množině $\Omega \subset \mathbb{R}^2$, jestliže $\exists\, U: \mathbb{R}^2 \to \mathbb{R}$ takové, že LHS je TD funkce U na Ω, neboli $\forall (x,y) \in \Omega$ a $(h_1,h_2) \in \mathbb{R}^2$ platí

$$dU(x,y)(h_1,h_2) = M(x,y)h_1 + N(x,y)h_2 .$$

Pak U nazýváme potenciálem rovnice.

Věta (ověření): Nechť U je potenciálem rovnice na oblasti $\Omega \subset \mathbb{R}^2$, $M,N \in C(\Omega)$ a $N \neq 0$ na Ω. Pak každým bodem $(x_0,y_0) \in \Omega$ prochází právě jedno řešení rovnice $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ a je implicitně dáno vztahem $U(x,y) = U(x_0,y_0)$.

Pokud $M \neq 0$ na Ω $\to$ $M(x,y)\frac{dx}{dy} + N(x,y) = 0$.

$\to$ Podmínka pro existenci potenciálu: $\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x} \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\to$ Pokud je splněna $\to$ hotovo.

$\to$ Pokud není splněna, můžeme zkusit nalézt integrační faktor.

$$\mu(x,y) = m(\varphi(x,y)): \quad \frac{\partial}{\partial y}\big(\mu(x,y)M(x,y)\big) = \frac{\partial}{\partial x}\big(\mu(x,y)N(x,y)\big)$$

$$\to m'(\varphi(x,y))\frac{\partial \varphi}{\partial y}M + m(\varphi(x,y))\frac{\partial M}{\partial y} = m'(\varphi(x,y))\frac{\partial \varphi}{\partial x}N + m(\varphi(x,y))\frac{\partial N}{\partial x}$$

$$\to \frac{m'(\varphi(x,y))}{m(\varphi(x,y))} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M\frac{\partial \varphi}{\partial y} - N\frac{\partial \varphi}{\partial x}} =: \psi(x,y) \overset{?}{=} H(\varphi(x,y))$$

if so

$$\frac{m'(z)}{m(z)} = H(z)$$

$$\Rightarrow m(z) = e^{\int H(z)\,dz}$$

$\varphi(x,y)$ zkusíme nastřelit:

$\varphi(x,y) = x \;\vee\; \varphi(x,y) = y \;\vee\; xy \;\vee\; x+y$

(12) $\underbrace{2xy}_{M(x,y)}\,dx + \underbrace{(x^2-y^2)}_{N(x,y)}\,dy = 0$

- $\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$ ✓ $\Rightarrow$ Je to rovnice ve tvaru TD.

- $M = 2xy = \frac{\partial U}{\partial x} \Rightarrow$

$\Rightarrow U(x,y) = \int 2xy\,dx = x^2y + C(y)$

zafixovaný $y \in \mathbb{R}$

- Také $N = x^2 - y^2 = \frac{\partial U}{\partial y} = x^2 + C'(y)$

$C'(y) = -y^2$

$C(y) = \int -y^2\,dy = -\frac{y^3}{3} + C$

$\Rightarrow U(x,y) = x^2y - \frac{y^3}{3} + C$ ... Potenciál, že $dU(x,y)(h_1,h_2) = 2xy\,h_1 + (x^2-y^2)h_2$

\+ poč. podmínka $y(x_0) = y_0$ (nebo $x(y_0) = x_0$) $\Rightarrow$ Řešení $U(x,y) = U(x_0,y_0)$.

(13) $\underbrace{e^{-y}\,dx}_{M(x,y)} \underbrace{- (2y + xe^{-y})}_{N(x,y)}\,dy = 0$

- TD?

$$\frac{\partial M}{\partial y} = -e^{-y} \qquad \frac{\partial N}{\partial x} = -e^{-y} \qquad \checkmark$$

- $M = e^{-y} = \frac{\partial U}{\partial x} \Rightarrow U(x,y) = \int e^{-y}\,dx = xe^{-y} + C(y)$

- $N = -(2y + \cancel{xe^{-y}}) = \frac{\partial U}{\partial y} = -\cancel{xe^{-y}} + C'(y)$

$$C'(y) = -2y$$

$$C(y) = \int -2y\,dy = -y^2 + c$$

$\Rightarrow U(x,y) = +xe^{-y} - y^2 + c$ ... Potenciál

$$dU(x,y)(dx,dy) = +e^{-y}dx - (2y + xe^{-y})dy$$

$M(x,y) = e^{-y} \neq 0 \;\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow \frac{dx}{dy} e^{-y} - (2y + xe^{-y}) = 0$ má řešení

$$xe^{-y} - y^2 + c = \underbrace{U(x_0,y_0)}_{\text{poč. podm.}}$$

$$\underline{x = \left(y^2 + U(x_0,y_0) - c\right)e^{y}}$$

(14) $\underbrace{\dfrac{3x^2+y^2}{y^2}}_{M(x,y)}\,dx - \underbrace{\dfrac{2x^3+5y}{y^3}}_{N(x,y)}\,dy = 0 \qquad y \neq 0$

• TD? $\dfrac{\partial M}{\partial y} = -\dfrac{6x^2}{y^3} \qquad \dfrac{\partial N}{\partial x} = -\dfrac{6x^2}{y^3}$ ✓

• $M = \dfrac{3x^2}{y^2} + 1 = \dfrac{\partial U(x,y)}{\partial x} \Rightarrow U(x,y) = \int \left(\dfrac{3x^2}{y^2} + 1\right) dx$

$= \dfrac{x^3}{y^2} + x + C(y)$

• $N = -\cancel{\dfrac{2x^3}{y^3}} - \dfrac{5}{y^2} = \dfrac{\partial U(x,y)}{\partial y} = -\cancel{\dfrac{2x^3}{y^3}} + C'(y)$

$\Rightarrow C'(y) = -\dfrac{5}{y^2}$

$C(y) = \int -\dfrac{5}{y^2}\,dy = \dfrac{5}{y} + c$

$\Rightarrow U(x,y) = \dfrac{x^3}{y^2} + x + \dfrac{5}{y} + c$ ... potenciál

$dU(x,y)(dx,dy) = \dfrac{3x^2+y^2}{y^2}\,dx - \dfrac{2x^3+5y}{y^3}\,dy$

$\Rightarrow$ Řešení rovnice: $\dfrac{x^3}{y^2} + x + \dfrac{5}{y} + c = \underbrace{U(x_0,y_0)}_{\text{poč. podm.}}$

(15) $\underbrace{(x^2+y)}_{M}dx \underbrace{- x}_{N}dy = 0 \qquad \mu = \mu(x)$

- Vstupy: $\frac{\partial M}{\partial y} = \underline{1} \neq \frac{\partial N}{\partial x} = \underline{-1}$ ... Není to rce ve tvaru TD

- Integrační faktor? ... $\mu = \mu(x)$

$$\frac{\mu'(\phi(x,y))}{\mu(\phi(x,y))} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M\frac{\partial \phi}{\partial y} - N\frac{\partial \phi}{\partial x}} \quad \& \quad \phi(x,y) = x \; : \; \frac{\mu'(\overbrace{\phi(x,y)}^{x})}{\mu(\phi(x,y))} = \frac{-1-1}{0+x} = \frac{-2}{x}$$

$$\frac{\mu'(x)}{\mu(x)} = -\frac{2}{x}$$

$\mu = \mu(x)$: $\frac{\partial}{\partial y}\left(\mu(x)(x^2+y)\right) = -\frac{\partial}{\partial x}\left(\mu(x)\, x\right)$

$$\mu(x)\cdot 1 = \mu(x)(-1) - x\mu'(x) \rightarrow \frac{\mu'(x)}{\mu(x)} = -\frac{2}{x}$$

$$\Rightarrow \ln|\mu(x)| = -2\ln|x| \rightarrow \mu(x) = \frac{1}{x^2}$$

$\rightarrow \underbrace{\left(1+\frac{y}{x^2}\right)}_{\tilde{M}}dx \underbrace{-\frac{1}{x}}_{\tilde{N}}dy = 0$ ... Rovnice ve tvaru tot. diferenciálu

- $\tilde{M} = 1 + \frac{y}{x^2} = \frac{\partial U}{\partial x} \rightarrow U(x,y) = \int\left(1+\frac{y}{x^2}\right)dx = x - \frac{y}{x} + c(y)$

- $\tilde{N} = \cancel{-\frac{1}{x}} = \frac{\partial U}{\partial y} = \cancel{-\frac{1}{x}} + c'(y) \rightarrow c'(y) = 0 \rightarrow c(y) = C$

$\Rightarrow U(x,y) = x - \frac{y}{x} + C$

Řešení $U(x,y) = U(x_0,y_0)$

(16) $\underbrace{(xy^2+y)}_{M}dx - \underbrace{x}_{N}dy = 0 \qquad \mu = \mu(y)$

$$\frac{\partial M}{\partial y} = 2xy + 1 \neq \frac{\partial N}{\partial x} = -1$$

$$\frac{\mu'(y)}{\mu(y)} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M\frac{\partial \varphi}{\partial y} - \cancel{N\frac{\partial \varphi}{\partial x}}} = \frac{-1-2xy-1}{xy^2+y} = -\frac{2\cancel{(1+xy)}}{y\cancel{(1+xy)}} = -\frac{2}{y}$$

$\varphi(x,y) = y$

$$\frac{\partial}{\partial y}\left(\mu(y)(xy^2+y)\right) = \frac{\partial}{\partial x}\left(\mu(y)(-x)\right) \rightsquigarrow \frac{\mu'(y)}{\mu(y)} = -\frac{2}{y}$$

$$\Rightarrow \mu(y) = \frac{1}{y^2}$$

$$\rightsquigarrow \underbrace{\left(x + \frac{1}{y}\right)}_{\tilde{M}}dx - \underbrace{\frac{x}{y^2}}_{\tilde{N}}dy = 0 \quad \dots \checkmark$$

- $\tilde{M} = x + \frac{1}{y} = \frac{\partial U}{\partial x} \rightarrow U(x,y) = \int\left(x+\frac{1}{y}\right)dx = \frac{x^2}{2} + \frac{x}{y} + c(y)$
- $\tilde{N} = -\cancel{\frac{x}{y^2}} = \frac{\partial U}{\partial y} = -\cancel{\frac{x}{y^2}} + c'(y) \rightarrow c'(y) = 0 \rightarrow c(y) = C$

$\rightsquigarrow$ Řešení ve tvaru $\frac{x^2}{2} + \frac{x}{y} = C$ ← z poč. podm.

(17) $(x^2+x^2y+2xy-y^2-y^3)dx + (y^2+xy^2+2xy-x^2-x^3)dy = 0$

$$\mu = \mu(x+y)$$

• $\dfrac{\partial M}{\partial y} = x^2+2x-2y-3y^2 \neq \dfrac{\partial N}{\partial x} = y^2+2y-2x-3x^2$

• $$\frac{\mu'(\phi(x,y))}{\mu(\phi(x,y))} = \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M\frac{\partial \phi}{\partial y}-N\frac{\partial \phi}{\partial x}} = \frac{y^2+\cancel{2y}-2x-3x^2-x^2-2x+\cancel{2y}+3y^2}{x^2+x^2y+\cancel{2xy}-y^2-y^3-y^2-xy^2-\cancel{2xy}+x^2+x^3}$$

$$\phi(x,y)=x+y$$

$$= \frac{-4y^2+4y-4x-4x^2}{2x^2+x^3-2y^2-y^3+x^2y-xy^2}$$

$$= \frac{4[(y^2-x^2)+(y-x)]}{(x-y)(x^2+2xy+y^2+2x+2y)}$$

$$= \frac{4(y-x)(x+y+1)}{(x-y)(x+y)(x+y+2)} = -4\,\frac{x+y+1}{(x+y)(x+y+2)}$$

-1

$$z \equiv x+y \Rightarrow \frac{\mu'(z)}{\mu(z)} = -4\,\frac{1+z}{z(z+2)}$$

$$\rightarrow \ln|\mu(z)| = \cancel{\int} -4\int \frac{1+z}{z(z+2)}\,dz = -2\int \frac{1}{z+2}\,dz - 2\int \frac{1}{z}\,dz$$

$$= -2\ln|z+2| - 2\ln|z| = \ln\frac{1}{(z+2)^2} + \ln\frac{1}{z^2}$$

$$= \ln\frac{1}{z^2(z+2)^2} \rightarrow \mu(z) = \frac{1}{z^2(z+2)^2}$$

$$\rightarrow \mu(x+y) = \frac{1}{(x+y)^2(x+y+2)^2}$$

...

(12) $x^2y^3 + y + (x^3y^2\cancel{y} - x)\,y' = 0$ $\qquad \mu = \mu(xy)$

$\rightarrow \underbrace{(x^2y^3+y)}_{M}dx + \underbrace{(x^3y^2-x)}_{N}dy = 0$

- $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ ✓

- $\dfrac{m'(\phi(x,y))}{m(\phi(x,y))} = \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M\frac{\partial \phi}{\partial y} - N\frac{\partial \phi}{\partial x}} = \dfrac{\cancel{3x^2y^2} - 1 - (\cancel{3x^2y^2} + 1)}{(\cancel{x^2y^3}+y)x - (\cancel{x^3y^2}-x)y} = \dfrac{-2}{2xy} = -\dfrac{1}{xy}$

$\phi(x,y) = xy \equiv z$

$\rightarrow \dfrac{m'(z)}{m(z)} = -\dfrac{1}{z} \rightarrow \ln|m(z)| = -\ln|z| \rightarrow m(z) = \dfrac{1}{z}$

$m(xy) = \dfrac{1}{xy}$

$\rightarrow \underbrace{\left(xy^2 + \frac{1}{x}\right)}_{\tilde{M}}dx + \underbrace{\left(x^2y - \frac{1}{y}\right)}_{\tilde{N}}dy = 0$

- $\dfrac{\partial \tilde{M}}{\partial y} = \dfrac{\partial \tilde{N}}{\partial x}$ ✓

- $\tilde{M} = xy^2 + \dfrac{1}{x} = \dfrac{\partial U}{\partial x} \rightarrow U(x,y) = \int\left(xy^2 + \frac{1}{x}\right)dx = \dfrac{x^2y^2}{2} + \ln|x| + c(y)$

- $\tilde{N} = \cancel{x^2y} - \dfrac{1}{y} = \dfrac{\partial U}{\partial y} = \cancel{\dfrac{x^2 2y}{2}} + c'(y) \rightarrow c'(y) = -\dfrac{1}{y}$

$c(y) = -\ln|y| + c$

$\rightarrow U(x,y) = \dfrac{x^2y^2}{2} + \ln|x| - \ln|y| + c$

$= \dfrac{x^2y^2}{2} + \ln\left|\dfrac{x}{y}\right| + c = U(x_0, y_0)$