Sada příkladů 2/10

Funkce více proměnných

Parciální derivace

V následujících příkladech zjistěte, kde jsou funkce definované, spojité, kde mají parciální derivace 1. řádu a kde jsou spojité 1. parciální derivace

1. $f(x, y) = \ln(x + y)$
2. $f(x, y, z) = \cos x \cosh y$
3. $f(x, y) = |x||y|$
4. $f(x, y) = \sqrt[3]{xy}$
5. $f(x, y) = \sqrt[5]{x^5 + y^5}$
6. $f(x, y, x) = x^{\frac{y}{z}}$.
7. Nechť $\alpha \in \mathbb{R}$. Pro jaké hodnoty α bude mít funkce

$$f(x, y) = (x^2 + y^2)^\alpha \sin \frac{1}{x^2 + y^2}$$

parciální derivace 1. řádu v bodě $(0, 0)$?

Spočtěte parciální derivace 2. řádu a zjistěte, zda jsou záměnné

8. $f(x, y) = x^4 + y^4 - 4x^2y^2$
9. $f(x, y) = \frac{x}{y^2}$
10. $f(x, y) = x \sin(x + y)$
11. $f(x, y) = \operatorname{tg} \frac{x^2}{y}$
12. $f(x, y, z) = x^{y^z}$
13. $f(x, y) = \operatorname{arctg} \frac{x+y}{1-xy}$

14. $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$ (Uvažujte bod (0,0).)

15. Spočtěte derivaci funkce $x^2 - y^2$ v bodě (1,1) ve směru jednotkového vektoru, který svírá s kladným směrem osy x úhel $\frac{\pi}{3}$.

16. Najděte jednotkový vektor, v jehož směru má derivace $x^2 - xy + y^2$ v bodě (1,1) největší, nejmenší a nulovou hodnotu.

17. Spočtěte $\frac{\partial F}{\partial u}$, kde $F = f(g)$, $f(x,y,z)$ je daná funkce a $g_1(u,v) = (u^2-1)/2v$, $g_2(u,v) = (u+v)/(u-v)$, $g_3(u,v) = u^2 - v^2$.

18. Nechť $f(s,t)$ je hladká nezáporná funkce na $\mathbb{R}^2$. Vyjádřete parciální derivace 1. řádu funkce $g(x,y) = f(x,y)^{f(y,x)}$ pomocí hodnot f a jejich parciálních derivací.

Funkce více proměnných – parciální derivace Počty 2/10 + 2/11

Parciální derivace

$$\frac{\partial f}{\partial x_i}(x_1,\dots,x_N) \equiv \lim_{h\to 0} \frac{f(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_N)-f(x_1,\dots,x_N)}{h}$$

Derivace ve směru

ANO, pokud má TD

$$\frac{\partial f}{\partial \vec{v}} = \lim_{t\to 0}\frac{f(\vec{x}+t\vec{v})-f(\vec{x})}{t} \stackrel{?}{=} \sum_{i=1}^{N} v_i \frac{\partial f}{\partial x_i}$$

Totální diferenciál

$f(x)$ má v x_0 TD $\stackrel{def.}{\iff} \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)-Lh}{\|h\|}=0$

$L = df(x_0)$ $\quad \left(\lim_{x\to x_0}\frac{f(x)-f(x_0)-L(x-x_0)}{x-x_0}=0\right)$

Věta 12.1.12 (TD ⇒ ...)

$f(x)$ má TD v x_0 ⇒ (i) $\exists \forall$ parc. derivace a $df(x_0)=\nabla f(x_0)$

(ii) $\exists$ derivace $\forall$ směrech a $\frac{\partial f}{\partial v}(x_0)=\nabla f(x_0)\cdot v$

(iii) f spojitá v x_0

Věta 12.1.14 (postačující podmínka TD)

- f má omezené parciální derivace na okolí x_0 ⇒ $f(x)$ spojitá v x_0
- f má na okolí x_0 parciální derivace a ty jsou spojité ⇒ má TD v x_0

Věta 12.1.17 (TD složeného zobrazení)

$f: \mathbb{R}^N \to \mathbb{R}^m$ má TD v $\underline{a}\in\mathbb{R}^N$ a $g: \mathbb{R}^m\to\mathbb{R}^k$ má TD v $f(a)$.

Potom $g\circ f$ má TD v $\underline{a}$ a $d(g\circ f) = dg(f(a))\circ df(a)$

$\left(d(g\circ f)(a)(h) = dg(f(a))(df(a)(h))\right) \ \forall h\in\mathbb{R}^N$

Věta 12.1.19 (Řetízkové pravidlo)

Předpoklad 12.1.17. Potom $\forall i\in\{1,\dots,N\}$: $\frac{\partial(g\circ f)}{\partial x_i}(a) = \sum_{j=1}^{m}\frac{\partial g}{\partial y_j}(f(a))\frac{\partial f_j}{\partial x_i}(a)$

Věta 12.2.2 (Záměnnost parc. der.)

$\Omega\subset\mathbb{R}^2$ otevřená množina a $f\in C^2(\Omega)$. Pak má f na Ω záměnné druhé parciální derivace.

Def (12.2.6) Totální diferenciál řádu k

$$d^k f(a)(h^1, \dots, h^k) = \sum \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)\, h^1_{i_1} \cdots h^k_{i_k}$$

Věta (12.2.4) Taylorův vzorec

$$f(a+h) = f(a) + \sum_{i_1=1}^{N} \frac{\partial f}{\partial x_{i_1}}(a)\, h_{i_1} + \frac{1}{2!} \sum_{i_1 i_2 = 1}^{N} \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(a)\, h_{i_1} h_{i_2}$$

$$+ \dots + \frac{1}{m!} \sum_{i_1 \dots i_m = 1}^{N} \frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}}(a)\, h_{i_1} \cdots h_{i_m}$$

$$+ \frac{1}{(m+1)!} \sum \frac{\partial^{m+1} f}{\partial x_{i_1} \cdots \partial x_{i_{m+1}}}(a + \theta h)\, h_{i_1} \cdots h_{i_{m+1}}$$

$\theta \in (0,1)$

Pokraj 2/10 – Funkce více proměnných, parciální derivace

+ v příkladech 1–7 uděláme i TD (sada 2/11)

(0) Jednoduchý příklad (12.4.1)

- $f(x,y) = x^2 + y^2$ ... paraboloid
  Def. na $\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

parc. der. 3. řádu a vyšší $= \phi$ $\quad \rightsquigarrow f \in C^\infty(\mathbb{R}^2)$

- $f(x,y) = x\sin(xy)$ $\quad \frac{\partial f}{\partial x} = \sin x + xy\cos(xy) \quad \frac{\partial f}{\partial y} = x^2\cos(xy)$

etc... $\rightsquigarrow f \in C^\infty(\mathbb{R}^2)$

- $f(xy) = (xy)^{1/3}$

... mechanicky: $\frac{\partial f}{\partial x} = \frac{1}{3} x^{-\frac{2}{3}} y^{1/3} \quad \forall x \neq 0$

def: $g: x \mapsto f(x,y)$ pro $y \in \mathbb{R}$ fixované

$\Rightarrow g'(0) = \lim_{x\to 0} \frac{1}{3} x^{-\frac{2}{3}} = \begin{cases} 0 & \text{pro } y = 0 \\ \operatorname{sgn}(y)\cdot\infty & \text{pro } y \neq 0 \end{cases}$

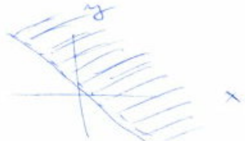
$\Rightarrow \frac{\partial f}{\partial x}(0,0) = 0$ & $\frac{\partial f}{\partial x}(0,y)$ neexistuje pro $y \neq 0$

Jinak: $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

$$\underset{y\neq 0}{\frac{\partial f}{\partial x}(0,y)} = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{(hy)^{1/3} - 0}{h} = y^{1/3}\cdot\infty$$

Příklad 1–7: Zjistěte definiční obor, spojitost, parciální derivace 1. řádu, spojitost 1. parc. derivací, totální diferenciál.

① $f(x,y) = \ln(x+y)$



• Def. pro $x+y>0$

• $\dfrac{\partial f}{\partial x} = \dfrac{1}{x+y} = \dfrac{\partial f}{\partial y}$ ... spojité na D_f $\Rightarrow$ má TD na D_f

$$Lh \equiv df(\vec{x})\cdot\underset{d\vec{x}}{\vec{h}} \equiv \nabla f\cdot h = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\cdot(h_1,h_2) = \frac{1}{x+y}(h_1+h_2) = \frac{h_1+h_2}{x+y}$$

② $f(x,y,z) = \cos x \cosh y$

• Def. na $\mathbb{R}^3$, spojité na D_f

• $\dfrac{\partial f}{\partial x} = -\sin x\cosh y$

$\dfrac{\partial f}{\partial y} = +\cos x\sinh y$ spojité na D_f

$\dfrac{\partial f}{\partial z} = 0$

$\Downarrow$ $\exists$ TD na $\mathbb{R}^3$

$$df(\vec{x})\cdot d\vec{x} = \nabla f\cdot(dx,dy,dz) = -\sin x\cosh y\,dx + \cos x\sinh y\,dy$$

③ $f(x,y) = |x||y|$

• Def. na $\mathbb{R}^2$, spojité na $\mathbb{R}^2 (D_f)$

• $\dfrac{\partial f}{\partial x} = |y|\operatorname{sgn} x$ pro $x\neq 0$ ... pro $x=0$: $\dfrac{\partial f}{\partial x} = \lim\limits_{h\to 0}\dfrac{|h||y| - 0|y|}{h} = \lim\limits_{h\to 0}|y|\dfrac{|h|}{h} = \Big<$ 0 pro $y=0$; neexistuje pro $y\neq 0$

$\Rightarrow \dfrac{\partial f}{\partial x}$ neexistuje v $(0,y)$ $y\neq 0$

$\Rightarrow \dfrac{\partial f}{\partial y}$ neexistuje v $(x,0)$ $x\neq 0$

v $(0,0)$: $\dfrac{\partial f}{\partial x} = 0 = \dfrac{\partial f}{\partial y}$

• TD mimo osy: $df(x,y)(h_1,h_2) = \operatorname{sgn} x\,|y|\,h_1 + \operatorname{sgn} y\,|x|\,h_2$

• TD na osách mimo počátek: neexistuje ($\Leftarrow$ neex. 1. parc. der.)

• TD v počátku: z definice. Pokud $\exists$, je to nulové zobrazení (neboť $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$)

$$\lim_{h\to 0}\frac{f(h_1,h_2) - f(0,0) - Lh}{\|h\|} \overset{?}{=} 0$$

$$0 \leq \lim\frac{|h_1||h_2| - 0 - 0}{\sqrt{h_1^2+h_2^2}} \leq \lim\frac{\frac{1}{2}(h_1^2+h_2^2)}{\sqrt{h_1^2+h_2^2}} = \frac{1}{2}\lim_{h\to 0}\sqrt{h_1^2+h_2^2} = \frac{1}{2}\lim_{h\to 0}\|h\| = 0$$

$\exists$ TD, je to nulové zobrazení

④ $f(x,y) = \sqrt[3]{xy}$

- $D_f = \mathbb{R}^2$, spojitá na D_f
- $\frac{\partial f}{\partial x} = \frac{1}{3}\frac{\sqrt[3]{y}}{x^{2/3}}$ pro $x \neq 0$ v $(0,y)$: $\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h} = \lim \frac{\sqrt[3]{y}}{h^{2/3}}$ $\begin{cases} = 0 & y=0 \\ = \nexists & y\neq 0 \end{cases}$

$\Rightarrow \frac{\partial f}{\partial x}(x,y) = \frac{1}{3}\sqrt[3]{\frac{y}{x^2}}$ pro $x\neq 0$, $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial x}(0,y)$ neexistuje pro $y \neq 0$

- $\Rightarrow \frac{\partial f}{\partial y}(x,y) = \frac{1}{3}\sqrt[3]{\frac{x}{y^2}}$ pro $y \neq 0$, $\frac{\partial f}{\partial y}(0,0) = 0$, $\frac{\partial f}{\partial y}(x,0)$ neexistuje pro $x \neq 0$

- TD mimo osy: $df(x,y)\vec{h} = \frac{1}{3}\sqrt[3]{\frac{y}{x^2}}\, h_1 + \frac{1}{3}\sqrt[3]{\frac{x}{y^2}}\, h_2$

TD na osách mimo počátek: neexistuje

TD v $(0,0)$: if $\exists$ $\Rightarrow$ nulové zobrazení.

$$\lim_{h\to 0} \frac{f(h_1,h_2) - f(0,0) - Lh}{\|h\|} = \lim_{h\to 0} \frac{\sqrt[3]{h_1 h_2} - 0 - 0}{\underbrace{\|h\|}_{\sqrt{h_1^2+h_2^2}}} \overset{h_1=h_2=h}{\sim} \lim \frac{h^{2/3}}{h} = \infty \neq 0$$

$\Rightarrow$ TD v $(0,0)$ neexistuje

⑤ $f(x,y) = \sqrt[5]{x^5+y^5}$

- $D_f = \mathbb{R}^2$, spojitá na D_f
- $\frac{\partial f}{\partial x} = \frac{1}{5}\frac{5x^4}{(x^5+y^5)^{4/5}} = \frac{x^4}{(x^5+y^5)^{4/5}}$
- $\frac{\partial f}{\partial y} = \ldots = \frac{y^4}{(x^5+y^5)^{4/5}}$

pro $x \neq -y$ spojité ... Pro $x+y=0$, $x\neq 0$ 1. parc. derivace neexistují

- v $(0,0)$: $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \frac{\sqrt[5]{h^5} - 0}{h} = 1 \rightarrow \frac{\partial f}{\partial x}(0,0) = 1 = \frac{\partial f}{\partial y}(0,0)$

- TD mimo $x+y=0$: $df(x,y)(h_1,h_2) = \frac{x^4 h_1 + y^4 h_2}{(x^5+y^5)^{4/5}}$

TD na $x+y=0$ mimo $(0,0)$: TD neexistuje

TD v $(0,0)$: if $\exists$, $Lh = h_1 + h_2$

$$\lim_{h\to 0} \frac{f(h_1,h_2) - f(0,0) - Lh}{\|h\|} = \lim_{h\to 0} \frac{\sqrt[5]{h_1^5+h_2^5} - h_1 - h_2}{\sqrt{h_1^2+h_2^2}}$$

$$\overset{h_1=h_2}{\sim} \lim_{h\to 0} \frac{\sqrt[5]{2h^5} - 2h}{\sqrt[2]{2h^2}} = \lim \frac{h(\sqrt[5]{2}-2)}{h\,\sqrt[2]{2}} \neq 0$$

$\Rightarrow$ TD v $(0,0)$ neexistuje

⑥ $f(x,y,z) = x^{\frac{y}{z}}$

• Definovaná pro $(x>0 \ \& \ z\neq 0)$ a $(x=0 \ \& \ yz>0 \ \& \ z\neq 0)$

$$\left.\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{y}{z}\, x^{\frac{y}{z}-1} = \frac{y}{xz}\, x^{\frac{y}{z}} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(e^{\frac{y}{z}\ln x}\right) = \frac{\ln x}{z}\, e^{\frac{y}{z}\ln x} = \frac{\ln x}{z}\, x^{\frac{y}{z}} \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left(\qquad \right) = -\frac{y\ln x}{z^2}\, x^{\frac{y}{z}}
\end{aligned}\right\} \ \exists \text{ pro } x>0 \ \& \ z\neq 0 \ (\text{a spojité}$$

• $x=0 \ \& \ xz>0 \ \& \ z\neq 0$: $\displaystyle \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(h,y,z)-f(0,y,z)}{h} = \lim_{h\to 0}\frac{h^{\frac{y}{z}}-0}{h} = \lim_{h\to 0} h^{\frac{y}{z}-1}$

$$= \begin{cases} 0 & \text{if } \frac{y}{z}-1>0 \\ \infty & \text{if } \frac{y}{z}-1<0 \\ 1 & \text{if } \frac{y}{z}-1=0 \end{cases}$$

• TD pro $x>0 \ \& \ z\neq 0$

$$df(x,y,z)(h_1,h_2,h_3) = e^{\frac{y}{z}}\left(\frac{y h_1}{xz} + \frac{h_2 \ln x}{z} - \frac{h_3 y \ln x}{z^2}\right)$$

• TD pro $x=0 \ \& \ xz>0 \ \& \ z\neq 0$ $\begin{cases} \& \ \frac{y}{z}-1>0 \ \ldots \\ \& \ \frac{y}{z}-1=0 \ \ldots \end{cases}$

(7) $\alpha \in \mathbb{R}$ ... Pro jaké hodnoty bude mít funkce $f(x,y) = (x^2+y^2)^\alpha \sin \frac{1}{x^2+y^2}$ parciální derivace 1. řádu v bodě $(0,0)$? Jsou 2. parc. der. záměnné? Co TD v $(0,0)$?

• $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$ $\Rightarrow$ Dodefinovat limitou v $(0,0)$:

$$f(0,0) := \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

pro $\alpha > 0$

• $$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{(h^2+0^2)^\alpha \sin\frac{1}{h^2+0^2}}{h} = \lim_{h\to 0} \frac{h^{2\alpha}\sin\frac{1}{h^2}}{h}$$

$$= \lim_{h\to 0} h^{2\alpha-1} \sin\frac{1}{h^2} = 0$$

pro $2\alpha - 1 > 0 \Leftrightarrow \alpha > \frac{1}{2}$

$$\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0) \quad \text{pro} \quad \alpha > \frac{1}{2}$$

• TD v $(0,0)$: Možné jen pro $\alpha > \frac{1}{2}$, kdy Lh bude nulové zobrazení

$$\lim_{h\to 0} \frac{f(h_1,h_2) - f(0,0) - Lh}{\|h\|} = \lim_{h\to 0} \frac{(h_1^2+h_2^2)^\alpha \sin\frac{1}{h_1^2+h_2^2}}{(h_1^2+h_2^2)^{1/2}}$$

$$= \lim_{h\to 0} \Big(\underbrace{\sqrt{h_1^2+h_2^2}}_{\|h\|}\Big)^{2\alpha-1} \sin\frac{1}{h_1^2+h_2^2} = 0 \checkmark$$

pro $\alpha > \frac{1}{2}$

Má TD v $(0,0)$ pro $\alpha > \frac{1}{2}$

• Záměnnost $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$?

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = 0 \checkmark$$

0

$$\frac{\partial f}{\partial x}(0,h) = \alpha (x^2+y^2)^{\alpha-1} (2x) \sin\frac{1}{x^2+y^2} + (x^2+y^2)^\alpha \cos\frac{1}{x^2+y^2} \cdot \frac{-1}{(x^2+y^2)^2} (2x) \Big|_{(x,y)=(0,h)}$$

$$= 0 + 0 = 0$$

Příklady 8 – 14: Spočtěte 2. parc. derivace a zjistěte, zda jsou záměnné.

(8) $f(x,y) = x^4 + y^4 - 4x^2y^2 \quad \longrightarrow \quad D_f = \mathbb{R}^2$, spojité na D_f

$$\frac{\partial f}{\partial x} = 4x^3 - 8xy^2 \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = -16xy \qquad \frac{\partial^2 f}{\partial x^2} = 12x^2 - 8y^2$$

$$\frac{\partial f}{\partial y} = 4y^3 - 8x^2y \qquad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = -16xy \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2 - 8x^2$$

Neboť f je $C^2(\mathbb{R}^2)$

(9) $f(x,y) = \frac{x}{y^2}$ ... def. pro $y \neq 0$, $C^2(D_f)$

$$\frac{\partial f}{\partial x} = \frac{1}{y^2} \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = -\frac{2}{y^3} \qquad \frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial f}{\partial y} = -\frac{2x}{y^3} \qquad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = -\frac{2}{y^3} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{6x}{y^4}$$

(10) $f(x,y) = x\sin(x+y)$ ... def. na $\mathbb{R}^2$, $C^2(D_f)$

$$\frac{\partial f}{\partial x} = \sin(x+y) + x\cos(x+y) \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \cos(x+y) - x\sin(x+y)$$

$$\frac{\partial f}{\partial y} = x\cos(x+y) \qquad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \cos(x+y) - x\sin(x+y)$$

(11) $f(x,y) = \tan\frac{x^2}{y}$ Def. pro $\frac{x^2}{y} \neq \frac{\pi}{2} + k\pi$ & $y \neq 0$

$$\frac{\partial f}{\partial x} = \frac{\frac{2x}{y}}{\cos^2\left(\frac{x^2}{y}\right)} \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = -\frac{\frac{2x}{y^2}}{\cos^2\left(\frac{x^2}{y}\right)} - \frac{\frac{4x}{y}}{\cos^3\left(\frac{x^2}{y}\right)}\left(-\sin\frac{x^2}{y}\right)\left(-\frac{x^2}{y^2}\right)$$

$$= -\frac{\frac{2x}{y^2}}{\cos^2\left(\frac{x^2}{y}\right)} - \frac{4x^3}{y^3}\,\frac{\tan\left(\frac{x^2}{y}\right)}{\cos^2\left(\frac{x^2}{y}\right)}$$

$$\frac{\partial f}{\partial y} = \frac{-\frac{x^2}{y^2}}{\cos^2\frac{x^2}{y}} \qquad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = -\frac{\frac{2x}{y^2}}{\cos^2\left(\frac{x^2}{y}\right)} - \frac{(-2)\frac{x^2}{y^2}}{\cos^3\frac{x^2}{y}}\left(-\sin\frac{x^2}{y}\right)\frac{2x}{y}$$

$$= -\frac{\frac{2x}{y^2}}{\cos^2\left(\frac{x^2}{y}\right)} - \frac{4x^3}{y^3}\,\frac{\tan\left(\frac{x^2}{y}\right)}{\cos^2\left(\frac{x^2}{y}\right)}$$

(12) $f(x,y,z) = x^{y^z}$ Def. pro $(x>0 \,\&\, y>0) \vee (x>0 \,\&\, y=0 \,\&\, z>0)$

$\vee (x=0 \,\&\, y^z>0 \,\&\, y>0) \vee (x=0 \,\&\, y^z>0 \,\&\, y=0 \,\&\, z>0)$

$$\frac{\partial f}{\partial x} = \frac{y^z}{x} x^{y^z} \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{z y^{z-1}}{x} x^{y^z}\left(\ln(x) y^z + 1\right) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} = z \ln x \, y^{z-1} x^{y^z} \qquad \frac{\partial}{\partial z}\frac{\partial f}{\partial y} = \ln(x) y^{z-1} x^{y^z} + z\ln(x) y^{z-1} \ln(y) x^{y^z}$$

$$+ z\ln^2(x) y^{2z-1} \ln(y) x^{y^z} = \frac{\partial}{\partial y}\frac{\partial f}{\partial z}$$

$$\frac{\partial f}{\partial z} = \ln x \, y^z \ln y \, x^{y^z}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{y^z \ln y}{x} x^{y^z}\left(\ln(x) y^z + 1\right) = \frac{\partial}{\partial z}\frac{\partial f}{\partial x}$$

(13) $f(x,y) = \operatorname{arctan} \frac{x+y}{1-xy}$ Def. pro $xy \neq 1$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+1} \qquad \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = \frac{1}{y^2+1} \qquad \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = 0$$

$$= \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{1-xy - (x+y)(-y)}{(1-xy)^2}$$

$$= \frac{\cancel{(1-xy)^2}}{(1-xy)^2+(x+y)^2} \cdot \frac{1-\cancel{xy}+\cancel{xy}+y^2}{\cancel{(1-xy)^2}}$$

$$= \frac{1+y^2}{1-\cancel{2xy}+x^2y^2+x^2+\cancel{2xy}+y^2} = \frac{\cancel{1+y^2}}{(\cancel{1+y^2})(1+x^2)} = \frac{1}{1+x^2}$$

(14) $$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Check: $$\lim_{(x,y)\to(0,0)} f(x,y) \underset{x,y \to r,\varphi}{=} \lim_{r\to 0+} r^2 \cos\varphi \sin\varphi \frac{\cancel{r^2}(\cos^2\varphi - \sin^2\varphi)}{\cancel{r^2}} = \lim_{r\to 0+} r^2 \cos\varphi \sin\varphi \cos(2\varphi) = 0 \checkmark$$

$\dfrac{\partial f}{\partial x}$:

$(x,y) \neq (0,0)$: $$\frac{y(3x^2-y^2)(x^2+y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$(x,y) = (0,0)$: $$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\cancel{h}\cdot 0\, \frac{h^2-0}{h^2+0} - \overbrace{0}^{f(0,0)}}{\cancel{h}} = 0$$

$\dfrac{\partial f}{\partial y}$:

$(x,y) \neq (0,0)$: $$\frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$(x,y) = (0,0)$: 0

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{h\to 0} \frac{\overbrace{\frac{\partial f}{\partial x}(0,h)}^{-\frac{h^5}{h^4}} - \overbrace{\frac{\partial f}{\partial x}(0,0)}^{0}}{h} = \lim_{h\to 0} \frac{-h-0}{h} = \underline{-1}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{h\to 0} \frac{\overbrace{\frac{\partial f}{\partial y}(h,0)}^{\frac{h^5}{h^4}} - \overbrace{\frac{\partial f}{\partial x}(0,0)}^{0}}{h} = \lim_{h\to 0} \frac{h-0}{h} = \underline{+1}$$

X

$\rightarrow$ Smíšené parciální derivace se nerovnají.

(15) Spočtěte derivaci funkce x^2-y^2 v bodě $(1,1)$ ve směru jednotkového vektoru, který svírá ~~...~~ s kladným směrem osy x úhel $\frac{\pi}{3}$

$\frac{\pi}{3} = 60^\circ$

$\nearrow$ $\vec{v} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ ... jednotkový vektor $\left(= (v_x, v_y)\right)$

$\vec{x} = (x,y)$ ... bod

$$\frac{\partial f}{\partial v} = \lim_{t \to 0} \frac{f(\vec{x} + t\vec{v}) - f(\vec{x})}{t}$$

- $f(\vec{x}) = f(1,1) = 1^2 - 1^2 = \underline{0}$
- $f(\vec{x} + t\vec{v}) = f\left(1 + \frac{t}{2}, 1 + \frac{\sqrt{3}t}{2}\right) = \left(1 + \frac{t}{2}\right)^2 - \left(1 + \frac{\sqrt{3}t}{2}\right)^2$

$$= \cancel{1} + \underline{t} + \frac{t^2}{4} - \cancel{1} - \underline{\sqrt{3}t} - \frac{3}{4}t^2 = (1-\sqrt{3})t - \frac{t^2}{2}$$

$$\frac{\partial f(1,1)}{\partial v} = \lim_{t \to 0} \frac{(1-\sqrt{3})t - \frac{t^2}{2} - 0}{t} = \underline{1 - \sqrt{3}}$$

z definice $\uparrow$

z věty (f má TD v a $\Rightarrow$ $\frac{\partial f}{\partial v}(a) = \nabla f(a) \cdot v$)

- $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ ... $\nabla f(x,y) = (2x, -2y)$ ... $\nabla f(1,1) = (2,-2)$
- $\frac{\partial f}{\partial v}(1,1) = \nabla f(1,1) \cdot (v_x, v_y) = (2,-2) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \underline{1 - \sqrt{3}}$

Ovšem je třeba ukázat, že $f(x,y)$ má v $(1,1)$ TD ... ano ... je tam C^2.

(16) Najděte jednotkový vektor, v jehož směru má derivace $f(x,y) = x^2 - xy + y^2$ v bodě $(1,1)$ největší, nejmenší a nulovou hodnotu.

- Maximální derivace je ve směru gradientu
- Minimální derivace je proti směru gradientu
- Nulová derivace je ve směrech kolmých na gradient.

$$\nabla f(1,1) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)_{(1,1)} = \left(2x - y, -x + 2y\right)_{(1,1)} = (1,1)$$

$$\to v_{max} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ ... jednotkový!}$$

$$v_{min} = -v_{max}$$

$$v_0 \text{ ... } v_0 \cdot \nabla f = 0 \text{ ... } v_0 \cdot v_{max} = 0 \Rightarrow v_0 = \pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

(17) Spočtěte $\frac{\partial F}{\partial u}$, kde $F = f(g)$, $f(x,y,z)$ je daná funkce a

$$g_1(u,v) = \frac{u^2-1}{2v} \qquad g_2(u,v) = \frac{u+v}{u-v} \qquad g_3(u,v) = u^2-v^2$$

- $F = f(g) = f \circ g \qquad f = f(x,y,z) \ \dots\ f: \mathbb{R}^3 \to \mathbb{R}$

$$g = g(u,v) = (g_1, g_2, g_3)(u,v) \ \dots\ f: \mathbb{R}^2 \to \mathbb{R}^3$$
$$= \left(g_1(u,v), g_2(u,v), g_3(u,v)\right)$$

$\to F = f(g) = f\left(g_1(u,v), g_2(u,v), g_3(u,v)\right)$

- Řetízkové pravidlo: $\frac{\partial (f \circ g)(a)}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial f}{\partial y_j}(f(a)) \frac{\partial f_j}{\partial x_i}(a)$

$$\to \frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial g_1}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial g_2}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial g_3}{\partial u}$$

$$= \frac{\partial f}{\partial x}\frac{2u}{2v} + \frac{\partial f}{\partial y}\frac{(\cancel{u}-v)-(\cancel{u}+v)}{(u-v)^2} + \frac{\partial f}{\partial z}2u$$

$$= \frac{u}{v}\frac{\partial f}{\partial x} - \frac{2v}{(u-v)^2}\frac{\partial f}{\partial y} + 2u\frac{\partial f}{\partial z}$$

$$= \left(\frac{u}{v}, -\frac{2v}{(u-v)^2}, 2u\right)\cdot \nabla f$$

(18) Nechť $f(s,t)$ je hladká nezáporná funkce na $\mathbb{R}^2$. Vyjádřete parc. der. 1. řádu funkce $g(x,y) = f(x,y)^{f(y,x)}$ pomocí hodnot f a jejích parciálních derivací.

$$g(x,y) = f(x,y)^{f(y,x)} = \exp\left[f(y,x) \ln f(x,y) \right]$$

$$\Rightarrow \frac{\partial g(x,y)}{\partial x} = \frac{\partial}{\partial x} \exp\left\{ f(y,x) \ln f(x,y) \right\}$$

$$= f(x,y)^{f(y,x)} \cdot \frac{\partial}{\partial x}\left\{ f(y,x) \cdot \ln f(x,y) \right\}$$

$$= f(x,y)^{f(y,x)} \left\{ \frac{\partial}{\partial x}\left[f(y,x) \right] \ln f(x,y) + f(y,x) \frac{\partial}{\partial x}\left[\ln f(x,y) \right] \right\}$$

$$= f(x,y)^{f(y,x)} \left\{ \underbrace{\frac{\partial f(u,v)}{\partial v}}_{\text{parc. der. podle 2. proměnné}}(y,x) \ln f(x,y) + \frac{f(y,x)}{f(x,y)} \underbrace{\frac{\partial f(u,v)}{\partial u}}_{\text{parc. der. podle 1. prom.}}(x,y) \right\}$$