Sada příkladů 2/7

Mocninné řady

Určete poloměr konvergence daných mocninných řad a vyšetřete konvergenci na kružnici konvergence ($z \in \mathbb{C}$)

1. $$\sum_{n=1}^{\infty} \frac{(z-3)^n}{n5^n}$$

2. $$\sum_{n=1}^{\infty} a^{n^2} z^n, \quad a \in \mathbb{R}^+$$

3. $$\sum_{n=1}^{\infty} \frac{a^n + b^n}{n} z^n, \quad a, b \in \mathbb{R}$$

4. $$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} (z-1)^n$$

5. $$\sum_{n=1}^{\infty} \frac{z^n}{n^p}, \quad p \in \mathbb{R}$$

6. $$\sum_{n=1}^{\infty} \frac{(2n)!!}{(2n+1)!!} z^n$$

   $(2n)!! = 2n(2n-2)(2n-4)\dots 4 \cdot 2,$
   $(2n+1)!! = (2n+1)(2n-1)(2n-3)\dots 3 \cdot 1$

7. $$\sum_{n=1}^{\infty} (-1)^n z^n \left(\frac{2^n (n!)^2}{(2n+1)!}\right)^p, \quad p \in \mathbb{R}.$$

8. Vyšetřete konvergenci zobecněné mocninné řady ($x \in \mathbb{R}$)

$$\sum_{n=1}^{\infty} n^2 \left(\frac{3x}{2+x^2}\right)^n.$$

Dokažte, že daná funkce je reálně analytická v počátku a nalezněte její Taylorovu řadu v nule, včetně intervalu konvergence

9. $\sin^2 x$

10. $\sqrt{1+x^2}$

11. $\displaystyle\int_0^x \frac{\operatorname{arctg} t}{t}\,\mathrm{d}t.$

Sečtěte funkční řady

12.
$$\sum_{n=1}^{\infty} n(n-1)x^{n-1}$$

13.
$$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}x^n.$$

Sečtěte číselné řady

14.
$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

15.
$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

16.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

Uvažujte $\operatorname{arctg} x$.

17.
$$\sum_{n=1}^{\infty} \frac{n}{(2n+1)!}$$

Uvažujte $(1+x)\mathrm{e}^{-x} - (1-x)\mathrm{e}^{x}$.

18.
$$\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

19.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}.$$

20. Nalezněte řešení Besselovy rovnice pro $n = 0$ ve tvaru $K_0(x) = \ln x \sum_{s=0}^{\infty} a_s x^s + \sum_{s=1}^{\infty} b_s x^s$.

21. Hledejte řešení Besselovy rovnice $x^2y'' + xy' + (x^2 - n^2)y = 0$ pro $n = \frac{1}{2}$ ve tvaru $x^\varrho \sum_{s=0}^{\infty} a_s x^s$ s vhodným ϱ.

Mocninné řady

~ zobecnění Taylorova rozvoje

Def: $\{a_n\} \subset \mathbb{C}$, $z_0 \in \mathbb{C}$, $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ ... mocninná řada se středem v z_0

(a_k … koeficienty mocninné řady)

$w = z - z_0 \rightarrow$ mocninná řada se středem v počátku

Věta (O konvergenci mocninné řady): $\{a_k\} \subset \mathbb{C}$, $R := \dfrac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$, $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$

(R … Poloměr konvergence mocninné řady)

(i) $\sum_0^\infty a_k z^k$ (AK) na B_R

(ii) $\sum_0^\infty a_k z^k$ nekonverguje na $\{z \in \mathbb{C} : |z| > R\}$

(iii) If $\exists \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = 1 = R$

(iv) If $\exists \lim_{k\to\infty} \sqrt[k]{|a_k|} = 1 = R$

Věta (Derivace mocninné řady): $\{a_k\} \subset \mathbb{C}$. Pak pro $x \in (-R, R)$, $R > 0$ je poloměr konvergence mocninné řady i platí $\left(\sum_0^\infty a_k x^k\right)' = \sum_1^\infty k a_k x^{k-1}$

Věta (Integrace mocninné řady): $\{a_k\} \subset \mathbb{C}$

(i) pro $x \in \mathbb{R}$ uvnitř konvergenčního kruhu $\int \left(\sum_0^\infty a_k x^k\right) dx = \sum_0^\infty \frac{a_k}{k+1} x^{k+1} + C$

(ii) Jestliže $a, b \in (-R, R)$

pak $(R)\int_a^b \left(\sum a_k x^k\right) dx = (N)\int_a^b \left(\sum a_k x^k\right) dx = \sum (N)\int_a^b a_k x^k dx = \sum (R)\int_a^b a_k x^k dx$

Def (Taylorova řada): $f: \mathbb{R} \to \mathbb{R}$ nekonečněkrát diferencovatelná v $x_0 \in \mathbb{R}$

Pak $\sum_0^\infty \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$ nazveme Taylorovou řadou f v x_0

Věta (Taylor – mocninná): $\{a_k\} \subset \mathbb{R}$, $x_0 \in \mathbb{R}$, $\exists \delta > 0$, že $\sum a_k (x-x_0)^k$ (K) na $(x_0-\delta, x_0+\delta)$

Pak je Taylorovou řadou svého součtu v bodě x_0.

Věta (Aritmetika Taylorových řad): $\{a_k\}, \{b_k\} \subset \mathbb{R}$, $f(x) = \sum_0^\infty a_k x^k$ a $g(x) = \sum_0^\infty b_k x^k$

obě (K) na okolí počátku. Pak $\exists$ okolí počátku, kde

(i) $(f+g)(x) = \sum_0^\infty (a_k + b_k) x^k$

(ii) $(fg)(x) = \sum_{k=0}^\infty \left(\sum_{m=0}^k a_m b_{k-m}\right) x^k$

(iii) if $a_0 = 0$, pak $g(f(x)) = \sum_{n=0}^\infty b_n \left(\sum_{k=0}^\infty a_k x^k\right)^n = \sum_{k=0}^\infty d_k x^k$, kde $d_k = \sum_{n=0}^\infty b_n \sum_{\substack{k_1, k_2, \dots, k_n \in \mathbb{N}_0 \\ k_1+k_2+\dots+k_n = k}} a_{k_1} a_{k_2} \cdots a_{k_n}$

Def (Reálně analytická funkce). Nechť $I \subset \mathbb{R}$ je interval a $f: I \to \mathbb{R}$. Řekneme, že f je reálně analytická na I, jestliže se dá na okolí každého bodu I vyjádřit Taylorovou řadou se středem v tomto bodě.

Věta (Abelova): $\{a_k\} \subset \mathbb{C}$ a příslušná mocninná řada $f(z) = \sum_{k=0}^{\infty} a_k z^k$ má poloměr konvergence $R \in (0,\infty)$. Je-li $\varphi \in [0,2\pi)$ takové, že pro $z = Re^{i\varphi}$ konverguje řada $\sum_{k=0}^{\infty} a_k z^k$, pak $t \to f(te^{i\varphi})$ je spojitá na $[0,R]$.

- Užití: Sečteme řadu uvnitř konv. kruhu $\Rightarrow$ dokonvergujeme na kružnici.

Lemma 6.8.15. *Nechť posloupnost* $\{a_n\} \subset \mathbb{R}$ *splňuje* $\lim_{n\to+\infty} |\frac{a_{n+1}}{a_n}| < 1$. *Pak* $\lim_{n\to+\infty} a_n = 0$.

Důkaz. Podle definice limity existují $q \in (0,1)$ a $n_0 \in \mathbb{N}$ taková, že

$$|a_{n+1}| \le q|a_n| \qquad \text{pro } n \ge n_0.$$

Odtud pro $n > n_0$

$$0 \le |a_n| \le q|a_{n-1}| \le q^2|a_{n-2}| \le \cdots \le q^{n-n_0}|a_{n_0}| \overset{n\to+\infty}{\to} 0.$$

□

Věta 6.8.16 (Základní Taylorovy rozvoje v počátku). *Platí (v levém sloupci vždy uvažujeme* $x \to 0$*):*

$$\mathrm{e}^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \mathrm{o}(x^n) \quad a \quad \mathrm{e}^x = \sum_{k=0}^{+\infty} \frac{x^k}{k!} \quad \forall x \in \mathbb{R}$$

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + \mathrm{o}(x^{2n+1}) \quad a \quad \cos x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \forall x \in \mathbb{R}$$

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+2}) \quad a \quad \sin x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \quad \forall x \in \mathbb{R}$$

$$\cosh x = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + \mathrm{o}(x^{2n+1}) \quad a \quad \cosh x = \sum_{k=0}^{+\infty} \frac{x^{2k}}{(2k)!} \quad \forall x \in \mathbb{R}$$

$$\sinh x = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + \mathrm{o}(x^{2n+2}) \quad a \quad \sinh x = \sum_{k=0}^{+\infty} \frac{x^{2k+1}}{(2k+1)!} \quad \forall x \in \mathbb{R}$$

$$\log(1+x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k} + \mathrm{o}(x^n) \quad a \quad \log(1+x) = \sum_{k=1}^{+\infty} (-1)^{k-1} \frac{x^k}{k} \quad \forall x \in (-1,1]$$

$$(1+x)^\alpha = \sum_{k=1}^{n} \binom{\alpha}{k} x^k + \mathrm{o}(x^n) \quad a \quad (1+x)^\alpha = \sum_{k=1}^{+\infty} \binom{\alpha}{k} x^k \quad \forall x \in (-1,1),$$

kde $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ *a zobecněné kombinační číslo* $\binom{\alpha}{k}$ *je definováno předpisem*

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}.$$

Důkaz. Výpočet derivací a dosazení $x_0 = 0$ pro získání Taylorových polynomů je jednoduchým cvičením. Budeme se zabývat jen odhadem velikosti zbytku a jeho konvergencí k nule pro $n \to +\infty$ na uvedených množinách.

Nejprve uvažme $f(x) = \mathrm{e}^x$. Pak pro Lagrangeův tvar zbytku máme

$$|R_{n+1}(x)| = \frac{\mathrm{e}^\xi |x|^{n+1}}{(n+1)!} \le \frac{\max\{\mathrm{e}^x, 1\}|x|^{n+1}}{(n+1)!}.$$

Pokrač 2/7 – Mocninné řady

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \quad (\text{if } \lim \exists)$$

① $\sum_{n=1}^{\infty} \frac{(z-3)^n}{n5^n}$ ... $\sum a_n (z-z_0)^n$ ← $z_0 = 3$ ... střed konvergence

koeficient $a_n = \frac{1}{n5^n}$

$$\frac{1}{R} \underset{\text{if } \exists}{=} \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n5^n}} = \lim_{n\to\infty} \frac{1}{5\sqrt[n]{n}} = \frac{1}{5} \Rightarrow R = 5$$

Poloměr konvergence $R = 5$: $|z-3| < 5 \Rightarrow$ AK

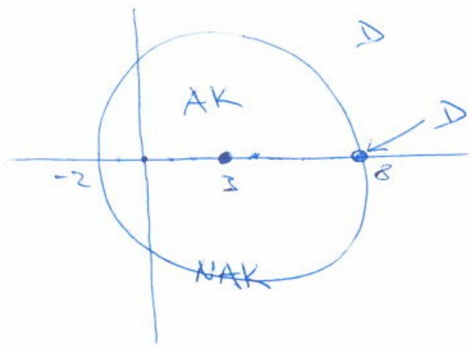
$|z-3| > 5 \Rightarrow$ D

$|z-3| = 5 \Rightarrow$ Nutno vyšetřit

$|z-3| = 5 \Leftrightarrow z-3 = 5e^{i\varphi}$ ⟶ Řada na hranici konv.: $\sum \frac{(5e^{i\varphi})^n}{n5^n} = \sum \frac{e^{in\varphi}}{n}$

$\hookrightarrow \sum \frac{e^{in\varphi}}{n}$

- $\varphi = 0$ ... $\sum \frac{1}{n}$ D
- $\varphi = \pi$ ... $\sum \frac{(-1)^n}{n}$ K Leibnitz
- $\varphi \neq 0, \pi$ ... $\sum \frac{e^{in\varphi}}{n} = \sum \frac{\cos(n\varphi) + i\sin(n\varphi)}{n}$ K Dirichlet



Abelova věta: $\forall \varphi \in (0, 2\pi)$ $\sum \frac{e^{in\varphi}}{n} = \lim_{x\to 5^-} \frac{1}{n}\left(\frac{x}{5}\right)^n e^{in\varphi}$

② $\sum_{n=1}^{\infty} a^{n^2} z^n$ $\quad \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim (a^{n\cdot n})^{\frac{1}{n}} = \lim_{n\to\infty} a^n$

a_n

$a \in \mathbb{R}^+$

$$\ldots = \begin{cases} 0 \text{ pro } a \leq 0 < 1 & \Rightarrow R = \infty \\ 1 \text{ pro } a = 1 & \Rightarrow R = 1 \\ \infty \text{ pro } a > 1 & \Rightarrow R = 0 \end{cases}$$

• $a=1 \Rightarrow R=1$, na komr. kružnici: $\sum e^{i\varphi n} = \sum (\cos(\varphi n) + i \sin(\varphi n))$

$\varphi = 0$ $\quad \sum 1$ .. Ⓓ

$\varphi \in (0, 2\pi)$: Ⓞ

③ $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{a^n+b^n}{n}}_{=:c_n} z^n \quad a,b \in \mathbb{R}$

BÚNO: $|a| \geq |b|$ (if $|a|<|b|$, přejmenujeme $a \leftrightarrow b$)

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \limsup \sqrt[n]{\left|\frac{a^n+b^n}{n}\right|} = \limsup \frac{\overbrace{\sqrt[n]{|a^n|}}^{\to |a|}\ \overbrace{\sqrt[n]{\left|1+\left(\frac{b}{a}\right)^n\right|}}^{\to 1}}{\underbrace{\sqrt[n]{n}}_{\to 1}} = |a|$$

$\Rightarrow R = \frac{1}{|a|}$

- $|z| < \frac{1}{|a|}$ ... AK
- $|z| > \frac{1}{|a|}$ ... D
- $|z| = \frac{1}{|a|}$ ... $z = \frac{1}{|a|^n} e^{i\varphi}$

$$\to \sum \frac{a^n+b^n}{n}\left(\frac{e^{i\varphi}}{|a|}\right)^n = \sum \frac{1}{n}\left(\frac{a}{|a|}\right)^n\left(1+\left(\frac{b}{a}\right)^n\right)e^{in\varphi}$$

• $|a| > |b|$

- $a>0 \to \sum \frac{1}{n}\left(1+\underbrace{\left(\frac{b}{a}\right)^n}_{\to 0}\right)e^{in\varphi} \sim \sum \frac{e^{in\varphi}}{n}$

  K pro $\varphi \neq 0$

- $a<0 \to \sum \frac{(-1)^n}{n}\left(1+\underbrace{\left(\frac{b}{a}\right)^n}_{\to 0}\right)e^{in\varphi} \sim \sum \frac{(-1)^n}{n} e^{in\varphi}$

  $(-1)^n e^{in\varphi} = \cos(n\pi)(\cos(n\varphi)+i\sin(n\varphi))$

  $\cos((\pi+\varphi)n) = \cos(n\pi)\cos(n\varphi) - \cancel{\sin(n\pi)}\sin(n\varphi)$

  $\sin((\pi+\varphi)n) = \cancel{\sin(n\pi)}\cos(n\varphi) + \cos(n\pi)\sin(n\varphi)$

  $\to \underbrace{\cos((\pi+\varphi)n) + i\sin((\pi+\varphi)n)}$

  omez. částeční souč. pro $\varphi \neq \pi$

  K pro $\varphi \neq \pi$

- $a=0 \Rightarrow b=0 \Rightarrow \sum 0$ ... K

• $|a| = |b|$

- $a=b>0 \Rightarrow \sum \frac{2}{n} e^{in\varphi}$ ... K pro $\varphi \neq 0$
- $a=b<0 \Rightarrow 2\sum \frac{(-1)^n}{n} e^{in\varphi}$ ... K pro $\varphi \neq \pi$
- $a=-b \Rightarrow \sum \frac{1+(-1)^n}{n} e^{in\varphi}$ ... K pro $\varphi \neq 0$ & $\varphi \neq \pi$

(4) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} (z-1)^n$

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \quad \Rightarrow \quad R = \frac{1}{e}$$

na ∂: $|z-1| = \frac{1}{e} \to z-1 = \frac{e^{i\varphi}}{e} \to \sum_{1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} e^{in\varphi}$

$$\lim \frac{\left(1+\frac{1}{n}\right)^{n^2}}{e^n} = \lim e^{n^2 \ln\left(1+\frac{1}{n}\right) - n} = e^{\lim\left(n^2 \ln\left(1+\frac{1}{n}\right) - n\right)}$$

$$= e^{\lim n^2\left(\cancel{\frac{1}{n}} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right) - \cancel{n}} = e^{\lim\left(-\frac{1}{2} + O\left(\frac{1}{n}\right)\right)} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} \neq 0$$

Není splněna nutná podmínka konvergence.
$\to$ Na ∂ nekonverguje.

(5) $\sum_{n=1}^{\infty} \frac{z^n}{n^p} \quad p \in \mathbb{R}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim \left(\frac{n+1}{n}\right)^p = \lim \left(1+\frac{1}{n}\right)^p = 1$$

- $|z| < 1$ (AK)
- $|z| > 1$ (D)
- $|z| = 1 \dots z = e^{i\varphi} \to \sum \frac{e^{in\varphi}}{n^p}$
  - (AK) pro $p > 1$
  - (K) pro $p > 0 \quad \varphi \neq 0$
  - (O) pro $p = 0$
  - (D) pro $p < 0$

⑥ $\sum_{n=1}^{\infty} \frac{(2n)!!}{(2n+1)!!} z^n = \sum \frac{2n\cdot(2n-2)\cdot\ldots\cdot4\cdot2}{(2n+1)(2n-1)\cdot\ldots\cdot5\cdot3\cdot1} z^n$

$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim \frac{(2n)!!\,(2n+3)!!}{(2n+1)!!\,(2n+2)!!} = \lim \frac{2n+3}{2n+2} = 1$

$\forall z$

$|z| < 1 \ldots$ (AK)

$|z| > 1 \ldots$ (D)

$|z| = 1 \ldots z = e^{i\varphi} \ldots \sum \frac{(2n)!!}{(2n+1)!!} e^{in\varphi}$

AK? $\sum \frac{(2n)!!}{(2n+1)!!}$ ... cf. příklad 22 ze sady 2/5

X

$$\frac{a_n}{a_{n+1}} = \frac{2n+3}{2n+2} = 1 + \frac{1}{2n+2}$$

Raabe: $n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+2} \xrightarrow{n\to\infty} \frac{1}{2} < 1 \Rightarrow D$

NAK? $\sum \frac{(2n)!!}{(2n+1)!!} e^{in\varphi}$ $\varphi \neq 0$. $e^{in\varphi}$ omez. částečné součty

$\rightarrow \frac{(2n)!!}{(2n+1)!!}$ jde monotónně $\rightarrow 0$?

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{2n+2} > 1 \ldots \text{monotonie} ✓$$

$\frac{(2n)!!}{(2n-1)!!} \sim \sqrt{\pi n} \Rightarrow \frac{(2n)!!}{(2n+1)!!} \sim \frac{\sqrt{\pi n}}{2n+1} = \frac{\sqrt{\pi}}{2\sqrt{n} + \frac{1}{\sqrt{n}}} \xrightarrow{n\to\infty} 0$ ✓

$\Rightarrow$ Pro $|z| = 1$, $\varphi \neq 0$ (NAK) podle ~~...~~ Dirichleta

- $(2n)!! = \underbrace{(2n)(2n-2)(2n-4)\cdots(2)}_{n \text{ součinitelů}} = 2^n n!$
- $(2n+1)! = (2n+1)!!\,(2n)!! \Rightarrow (2n+1)!! = \frac{(2n+1)!}{(2n)!!} \Rightarrow \frac{(2n)!!}{(2n+1)!!} = \frac{((2n)!!)^2}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!}$
- Stirling's approximation: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \rightsquigarrow \ldots \rightarrow \frac{(2n)!!}{(2n+1)!!} \sim \frac{1}{\sqrt{n}}$

$\left(\left(\frac{n}{e}\right)^n < \frac{n!}{e} < n\left(\frac{n}{e}\right)^n \right)$

⑦ $\sum_{n=1}^{\infty} (-1)^n z^n \left(\frac{2^n (n!)^2}{(2n+1)!} \right)^p \quad p \in \mathbb{R}$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2^n (n!)^2}{(2n+1)!} \frac{(2n+3)!}{2^{n+1}((n+1)!)^2} \right|^p = \lim \left(\frac{(2n+2)(2n+3)}{2(n+1)^2} \right)^p$$

if ∃

$$= \lim \left(\frac{4(n+1)(n+\frac{3}{2})}{2(n+1)^2} \right)^p = 2^p \quad \begin{cases} |z| < 2^p \ldots \text{AK} \\ |z| > 2^p \ldots \text{D} \end{cases}$$

$|z| = 2^p \ldots z = 2^p e^{i\varphi}$

$$\sum_1^{\infty} (-1)^n \left(\frac{2^n (n!)^2}{(2n+1)!} \right)^p 2^{pn} e^{in\varphi} = \sum_1 (-1)^n \underbrace{\left(\frac{4^n (n!)^2}{(2n+1)!} \right)}_{:= b_n}{}^p e^{in\varphi}$$

$\sum (-1)^n e^{in\varphi}$ má omez. částečné součty pro $\varphi + \pi \neq 2k\pi \iff \varphi \neq \pi$

Leibniz? ... $\sum_1^{\infty} (b_n)^p (-1)^n e^{in\varphi}$ kde $b_n = \frac{4^n (n!)^2}{(2n+1)!}$

$$\cancel{\lim_{n\to\infty} b_n = \lim \frac{4^n (n!)^2}{(2n+1)!} = \lim 4^n \frac{\overbrace{n(n-1)\ldots 2 \cdot 1}^{n \text{ členů}}}{\underbrace{(2n+1)(2n)(2n-1)\ldots(n+2)(n+1)}_{n+1 \text{ členů}}}}$$

$$\cancel{\leq \lim 4^n \left(\frac{1}{2}\right)^n \frac{1}{n+1} = \lim \frac{2^n}{n+1}}$$

- $$\frac{b_{n+1}}{b_n} = \frac{4^{n+1}((n+1)!)^2}{(2n+3)!} \frac{(2n+1)!}{4^n (n!)^2} = \frac{4(n+1)^2}{(2n+3)(2n+2)} = \frac{4n^2 + 8n + 4}{4n^2 + 10n + 6} < 1$$

b_n klesá monot.

- Stirlingova formula: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{4^n (n!)^2}{(2n+1)!} \sim \lim_{n\to\infty} \frac{4^n (2\pi n) \left(\frac{n}{e}\right)^{2n}}{\sqrt{2\pi} \sqrt{2n+1} \left(\frac{2n+1}{e}\right)^{2n+1}}$$

$$= \lim \frac{4^n \sqrt{2\pi}\, n^{2n+1} \frac{1}{e^{2n}}}{2^{2n+1} \sqrt{2n+1} \left(n+\frac{1}{2}\right)^{2n+1} \frac{1}{e^{2n+1}}} = \lim \frac{\sqrt{2\pi}\, e}{2} \frac{1}{\sqrt{2n+1}} \frac{1}{1+\frac{1}{2n}} = \underline{0}$$

$(b_n)^p \xrightarrow{n\to\infty} 0$ pro $p > 0$ $\Rightarrow$ pro $|z| = 2^p$ <u>NAK pro $p > 0$</u>

<u>AK na $|z| = 2^p$?</u> $$\left(\frac{b_n}{b_{n+1}} \right)^p = \left(\frac{(n+1)(n+\frac{3}{2})}{(n+1)^2} \right)^p = \left(\frac{n^2 + \frac{5}{2}n + \frac{3}{2}}{n^2+2n+1} \right)^p = \left(1 + \frac{\frac{n}{2} + \frac{1}{2}}{n^2+2n+1} \right)^p$$

$$= 1 + \frac{p}{2n} + O\left(\frac{1}{n^2}\right) \ldots \text{Raabe} \cdot \text{K pro } \frac{p}{2} > 1 \ldots p > 2$$

⑧ $\sum_{n=1}^{\infty} n^2 \left(\frac{3x}{2+x^2}\right)^n$ $x \in \mathbb{R}$ ... zobecněná mocninná řada

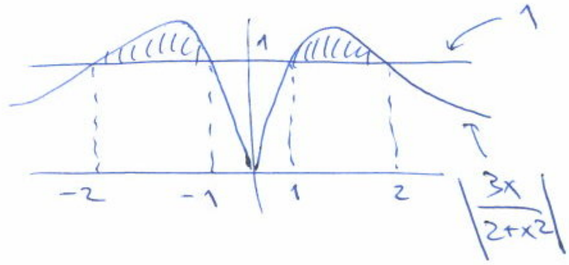
$\vdots$

$\sum_{1}^{\infty} n^2 y^n$, kde $y = \frac{3x}{2+x^2}$

$a_n = n^2 \leadsto \frac{1}{R} \underset{\text{if } \exists}{=} \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim \left(\sqrt[n]{n}\right)^2 = 1 \Rightarrow \underline{R = 1}$

AK pro $|y| < 1 \iff \left|\frac{3x}{2+x^2}\right| < 1$

$\iff x \in (-2,-1) \vee \cancel{x}$
$x \in (1,2)$



D pro $|y| > 1 \iff x \in (-\infty,-2) \cup (-1,1) \cup (2,\infty)$

pro $|y| = 1 \iff x = \pm 1, \pm 2$

$x = 1$ ... $\frac{3x}{2+x^2} = 1$ ... $\sum n^2 \longrightarrow$ D

$x = -1$ $= -1$ ... $\sum (-1)^n n^2 \longrightarrow$ D

$x = 2$ $= 1$

$x = -2$ $= -1$

9 $\sin^2 x$ ... Dokažte, že daná funkce je reálně analytická v počátku a nalezněte její Taylorovu řadu v nule, včetně intervalu konv.

$$\cos(2x) = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x)$$
$$= 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x$$

$$f(x) = \sin^2 x = \frac{1-\cos 2x}{2}$$

Víme: $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ ... Taylor $\cos x$ v $x=0$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$$

0. člen

$$\Rightarrow \sin^2 x = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \frac{1}{2}\sum_{1}^{\infty} (-1)^{n-1} \frac{(2x)^{2n}}{(2n)!}$$

$$\sin^2 x = \sum_{1}^{\infty} \frac{(-1)^{n-1}}{2} \frac{(2x)^{2n}}{(2n)!} = \sum_{1}^{\infty} \frac{(-1)^{n-1}}{2} \frac{(4x^2)^n}{(2n)!} \leftarrow 4x^2 =: z$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} = \lim_{n\to\infty} (2n+2)(2n+1) = \infty$$

$R = \infty$ ... AK $\forall\, z = 4x^2 \in \mathbb{C}$ $\quad x = \frac{\sqrt{z}}{2} \in \mathbb{C}$ $\quad$ AK $\forall x \in \mathbb{C}$

$\Rightarrow \forall x \in \mathbb{R}$

(10) $f = \sqrt{1+x^2}$ ... je analytická v $x=0$?

Zobecněná binomická věta : $(a+b)^\alpha = \sum_0^\infty \binom{\alpha}{k} a^{\alpha-k} b^k$

Taylor $\left(1+\frac{b}{a}\right)^\alpha$ v $\frac{b}{a}=0$ $\times$ a^α

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\binom{\alpha}{0} \equiv 1 \text{ (def.)}$$

$$\to (1+b)^\alpha = \sum_{k=0}^\infty \binom{\alpha}{k} b^k$$

$$\left(\tfrac{1}{2}-(k-1)\right) = \tfrac{1}{2}(1-2k+2) = -\tfrac{1}{2}(2k-3)$$

$$(1+x^2)^{\frac{1}{2}} = \sum_{k=0}^\infty \binom{\frac{1}{2}}{k} x^{2k}$$

$$\binom{\frac{1}{2}}{k} = \frac{\tfrac{1}{2}\overbrace{\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)\cdots\left(\tfrac{1}{2}-k+1\right)}^{k \text{ součinitelů}}}{k!}, \quad -\tfrac{1}{2},\ -\tfrac{3}{2}$$

$$= \frac{(-1)^{k-1}}{2^k}\frac{(2k-3)!!}{k!} \quad \leftarrow \text{od členu } k=2$$

$$\sqrt{1+x^2} = \sum_0^\infty \binom{\frac{1}{2}}{k} x^{2x} = \sum_0^\infty \frac{(-1)^{k-1}}{2^k}\frac{(2k-3)!!}{k!}(x^2)^k \quad (\text{Not clear for } k=0,1)$$

$$= \underset{\substack{\uparrow\\ k=0}}{1} + \underset{\substack{\uparrow\\ k=1}}{\frac{x^2}{2}} + \sum_2^\infty \underbrace{\frac{(-1)^{k-1}}{2^k}\frac{(2k-3)!!}{k!}}_{a_k}(x^2)^k$$

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{(2k-3)!!}{2^k\, k!}\,\frac{(k+1)!\,2^{k+1}}{(2k-1)!!} = \frac{2(k+1)}{(2k-1)} \xrightarrow{k\to\infty} 1 = R$$

AK pro $x^2 < 1$

$\Leftrightarrow$ $x \in (-1,1)$

(11) $\int_0^x \frac{\arctan t}{t}\, dt$ ... analytické v $x = 0$?

• Start with $(\arctan x)' = \underbrace{\frac{1}{1+x^2}}_{\text{součet geom. řady s } q=-x^2 \text{ a } 1.\text{člen} = 1} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_0^{\infty} (-1)^n (x^2)^n = \sum_0^{\infty} (-1)^n x^{2n}$

$\left|\frac{a_n}{a_{n+1}}\right| = 1 = R$

Konverg. pro $x^2 < 1 \iff x \in (-1,1)$

• $\arctan x = \int (\arctan x)' dx = \int \left(\sum_0^{\infty} (-1)^n x^{2n}\right) dx = \sum_0^{\infty} \int (-1)^n x^{2n} dx = \sum (-1)^n \frac{x^{2n+1}}{2n+1}$

stejný interval (K) $(-1,1)$

• $\frac{\arctan x}{x} = \frac{1}{x} \sum_0 (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_0 (-1)^n \frac{x^{2n}}{2n+1}$

• $\int_0^x \frac{\arctan t}{t} dt = \int_0^x \left(\sum_0^{\infty} (-1)^n \frac{t^{2n}}{2n+1}\right) dt = \sum_0^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{2n+1} dt$

$= \sum_0^{\infty} (-1)^n \frac{\left[t^{2n+1}\right]_0^x}{(2n+1)^2} = \sum_0^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2}$ ✓

Interval konvergence $x \in (-1,1)$

(12) Sečtěte řadu $\sum_{n=1}^{\infty} n(n-1)x^{n-1}$

⇝ Derivovat mocninnou řadu $\sum_0^{\infty} x^n = \frac{1}{1-x}$? ... $R = 1$

$$\left(\sum_0^{\infty} x^n\right)' = \sum_1^{\infty} n x^{n-1} = \left(\frac{1}{1-x}\right)'$$

$$\left(\sum_0^{\infty} x^n\right)'' = \left(\sum_1^{\infty} n x^{n-1}\right)' = \sum_2^{\infty} n(n-1)x^{n-2} = \sum_1^{\infty} n(n-1)x^{n-2} = \left(\frac{1}{1-x}\right)''$$

$$\sum_1^{\infty} n(n-1)x^{n-1} = x\left(\frac{1}{1-x}\right)'' = x\left(\frac{\cancel{0}(1-x)-1(-1)}{(1-x)^2}\right)' = x\left(\frac{1}{(1-x)^2}\right)'$$

$$= x\,\frac{-2(-1)}{(1-x)^3} = \frac{2x}{(1-x)^3} \quad \forall x \in (-1,1)$$

13 $\displaystyle\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n$ ... sečtěte funkční řadu

Note: $\displaystyle (1-x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x}} = \sum_{0}^{\infty} \binom{-\frac{1}{2}}{n} (-x)^n = \sum_{0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} (-1)^n x^n$

$$\binom{-\frac{1}{2}}{n} = \frac{\overbrace{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}^{n \text{ součinů}}}{n!} = \frac{(-1)^n \left(\frac{1}{2}\right)^n (2n-1)!!}{n!}$$

$$= (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

$$= \sum_{0}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n = 1 + \underbrace{\sum_{1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n}_{\text{to chceme spočíst}}$$

$$\to \sum_{1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n = \frac{1}{\sqrt{1-x}} - 1 \quad \text{pro } x \in (-1,1)$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(2n-1)!!}{(2n)!!} \frac{(2n+2)!!}{(2n+1)!!} = \frac{2n+2}{2n+1} \xrightarrow{n\to\infty} 1 = R$$

(14) Sečtěte číselnou řadu $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

$$\sum_{1}^{\infty} \frac{1}{n2^n} = \sum_{1}^{\infty} \frac{1}{n}\left(\frac{1}{2}\right)^n \quad \dots \quad \sum \frac{1}{n} x^n \quad \text{will } x = \frac{1}{2} \quad \text{ok} \checkmark$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n} \xrightarrow{n\to\infty} 1 = R$$

• Integrovat $\sum_{0}^{\infty} x^n = \frac{1}{1-x}$

$$\int\left(\sum_{0}^{\infty} x^n\right)dx = \sum_{0}^{\infty} \int x^n dx = \sum_{0}^{\infty} \frac{x^{n+1}}{n+1} \overset{n+1\to n}{=} \sum_{1}^{\infty} \frac{x^n}{n} = \int_0^x \frac{1}{1-t}\,dt \overset{x\in(-1,1)}{=} -\ln(1-x) + C$$

$$C = ? \quad \dots \quad x = 0: \quad \sum_{1}^{\infty} \frac{x^n}{n} = 0 \overset{!}{=} -\ln(1) + C = 0 + C \Rightarrow \underline{C = 0}$$

$$\Rightarrow \sum_{1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \quad \Rightarrow \sum_{1}^{\infty} \frac{1}{n2^n} = -\ln\frac{1}{2} = \underline{\ln 2}$$

(15) Sečtěte číselnou řadu $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

$$\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \underset{n=m+1}{=} \sum_{m=0}^{\infty} \frac{m+1}{m!} = \sum_{0}^{\infty} \frac{m}{m!} + \sum_{0}^{\infty} \frac{1}{m!}$$

• Víme: $e^x = \sum_{0}^{\infty} \frac{x^n}{n!}$ $\quad \left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)!}{n!} = n+1 \xrightarrow{n\to\infty} \underline{\infty = R}$ (AK) $\forall x \in \mathbb{R}$

$$\Rightarrow e^1 = \sum_{0}^{\infty} \frac{1}{n!} \quad \checkmark$$

• $$\sum_{0}^{\infty} \frac{m}{m!} = \sum_{1}^{\infty} \frac{m}{m!} = \sum_{1} \frac{1}{(m-1)!} \underset{m-1=n}{=} \sum_{0} \frac{1}{n!} = e$$

$$\Rightarrow \sum_{1}^{\infty} \frac{n^2}{n!} = e + e = \underline{2e}$$

(16) Sečtěte číselnou řadu $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$ Hint: Uvažujte arctan x

$$\sum_{1}^{\infty} \frac{(-1)^{n+1}}{2n-1} \underset{n=m+1}{=} \sum_{0}^{\infty} \frac{(-1)^{m+2}}{2m+1} = \sum_{0}^{\infty} \frac{(-1)^m}{2m+1} \ \dots \ \sum_{0}^{\infty} \frac{(-1)^m}{2m+1} x^m \ \text{pro } x = 1$$

$$\left| \frac{a_n}{a_{n+1}} \right| = \frac{2m+3}{2m+1} \xrightarrow{m \to \infty} 1 = R$$

Pozor, nutno ověřit (K)

• $(\arctan x)' = \frac{1}{1+x^2} = \sum_{0}^{\infty} (-x^2)^m = \sum (-1)^m x^{2m}$

• $\Rightarrow \arctan x = \int_0^x \sum_0^{\infty} (-1)^m t^{2m} \, dt = \sum_0^{\infty} \int_0^x (-1)^m t^{2m} \, dt = \sum_0^{\infty} (-1)^m \frac{x^{2m+1}}{2m+1} \equiv f(x)$

$R = 1 \longrightarrow$ na $(-1,1)$

$f(x) = \arctan x$ na $x \in (-1,1)$

pro $|x| > 1$ řada diverguje

pro $x = \pm 1$: Abelova věta

• $\varphi = 0 \Rightarrow z = e^{i0} = 1 \Rightarrow \sum_0^{\infty} (-1)^m \frac{1}{2m+1}$ ... (K) podle Leibnitze

$\Rightarrow x \to f(x)$ je spojitá na $[0,1]$

$\Rightarrow f(1) \equiv \sum_0^{\infty} (-1)^m \frac{1}{2m+1} = \arctan 1 = \frac{\pi}{4}$

$f(1) = \lim_{x \to 1-} f(x) = \lim_{x \to 1-} \arctan x = \frac{\pi}{4}$

(17) Sečtěte číselnou řadu $\sum_{n=1}^{\infty} \frac{n}{(2n+1)!}$ Hint: Uvažujte $(1+x)e^{-x} - (1-x)e^{x}$

Víme: $e^{x} = \sum_{0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-x} = \sum_{0}^{\infty} \frac{(-1)^n x^n}{n!}$ ... $R = \infty$

$$\Rightarrow \quad (1+x)e^{-x} = \sum_{0}^{\infty} (-1)^n \frac{x^n}{n!} + \sum (-1)^n \frac{x^{n+1}}{n!}$$

$$-(1-x)e^{x} = -\sum_{0}^{\infty} \frac{x^n}{n!} + \sum \frac{x^{n+1}}{n!}$$

odečtení členů $2n$ $\quad$ odečtení členů $2n+1$

$$\bullet \; (1+x)e^{-x} - (1-x)e^{x} = -2\sum_{0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} + 2\sum_{0}^{\infty} \frac{x^{2n+1}}{(2n)!} \underbrace{\frac{2n+1}{2n+1}}_{=1}$$

$$= 2\sum_{0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \left(\cancel{-1} + 2n + \cancel{1}\right)$$

$$= 4\sum_{0}^{\infty} \frac{n}{(2n+1)!} x^{2n+1} \equiv f(x) \quad \underline{\forall x \in \mathbb{R}}$$

$$\bullet \; f(1) = 4 \underbrace{\sum_{0}^{\infty} \frac{n}{(2n+1)!}}_{\text{to chceme}} = (1+1)e^{-1} - \cancel{(1-1)e^{1}} = \frac{2}{e}$$

$$\Rightarrow \sum_{0}^{\infty} \frac{n}{(2n+1)!} = \frac{1}{4}\frac{2}{e} = \underline{\frac{1}{2e}}$$

(12) Sečtěte číselnou řadu $\sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!}$

- Remember úloha 13: $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n = \frac{1}{\sqrt{1-x}} - 1$ pro $x \in (-1,1)$ ... $\underline{R=1}$

- Teď: $x = -1$ ... $\underbrace{\sum (-1)^n \frac{(2n-1)!!}{(2n)!!}}$ ... K podle Leibnitze

↳ víme (1/2, úloha

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{2n+1}} \xrightarrow{n\to\infty} 0 \text{ a klesá (monot.)}$$

- Abelova věta: ⇒) Řada (K) pro $x = -1$ ⇒ $f(-1) = \lim_{x\to -1^+} f(x)$
  ↑ na R

$$= \lim_{x\to -1^+} \frac{1}{\sqrt{1-x}} - 1 = \frac{1}{\sqrt{2}} - 1 = \frac{1-\sqrt{2}}{\sqrt{2}} = \underline{-\frac{\sqrt{2}-1}{\sqrt{2}}}$$

(15) Sečtěte číselnou řadu $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$ ... (K) podle Leibnize

$$\sum_{1}^{\infty} \frac{x^n}{n(n+1)} \quad \text{pro } x = -1$$

• $\sum_{0}^{\infty} x^n = \frac{1}{1-x}$ ... $R = 1$ Pozor!

$c = 0$ ($x = 0 \Rightarrow$ ...)

• $\int \sum_{0}^{\infty} x^n dx = \sum_{0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{1}^{\infty} \frac{x^n}{n} = \int_0^x \frac{dt}{1-t} \overset{t<1}{=} -\ln(1-x) + c$

• $\int \sum_{1}^{\infty} \frac{x^n}{n} dx = \sum_{1}^{\infty} \frac{x^{n+1}}{n(n+1)} = x \underbrace{\sum_{1}^{\infty} \frac{x^n}{n(n+1)}}_{f(x)} = -\int_0^x \ln(1-u)\,du \; \left| \begin{array}{c} 1-u = v \\ -du = dv \\ \int_0^x \to \int_1^{1-x} \end{array} \right|$

$$= \int_0^x \ln v \, dv = v \ln v \Big|_1^{1-x} - \int_1^{1-x} 1\, dv =$$

$$= (1-x)\ln(1-x) - \cancel{1 \ln 1} - (\cancel{1}-x) + \cancel{1} + d$$

$$= (1-x)\ln(1-x) + x + \cancel{d}$$

$x = 0$: $0 = 0 + d \Rightarrow d = 0$

$$f(x) = \frac{1}{x}\left((1-x)\ln(1-x) + x\right) = \frac{1-x}{x}\ln(1-x) + 1$$

• Abelova věta: $f(-1) = \lim_{x \to -1^+} f(x) = \frac{2}{-1}\ln 2 + 1 = 1 - 2\ln 2 = \underline{1 - \ln 4}$

(21) Hledejte řešení Besselovy rovnice $x^2y'' + xy' + (x^2-n^2)y = 0$ pro $n = \frac{1}{2}$ ve tvaru $x^\rho \sum_{s=0}^{\infty} a_s x^s$ s vhodným ρ.

$$y = x^\rho \sum_0 a_s x^s = \sum_0 a_s x^{s+\rho}$$

$$\cancel{y' = \rho x^{\rho-1} \sum_0 a_s x^s + x^\rho \sum s a_s x^{s-1}}$$

$$y' = \sum_0 (s+\rho) a_s x^{s+\rho-1} \Rightarrow xy' = \sum_0 (s+\rho) a_s x^{s+\rho}$$

$$y'' = \sum_0 (s+\rho)(s+\rho-1) a_s x^{s+\rho-2} \Rightarrow x^2 y'' = \sum_0 (s+\rho)(s+\rho-1) a_s x^{s+\rho}$$

ODR:

$$\sum_0 \left[(s+\rho)(s+\rho-1) + (s+\rho) - \frac{1}{4}\right] a_s x^{s+\rho} + \sum_2 a_{s-2} x^{s+\rho} = 0$$

$$\left(\rho(\rho-\cancel{1}) + \cancel{\rho} - \frac{1}{4}\right) a_0 x^\rho + \left[(1+\rho)\rho + 1 + \rho - \frac{1}{4}\right] a_1 x^{\rho+1}$$

$$+ \sum_2^{\infty} \left\{\left[(s+\rho)(s+\rho-1) + s + \rho - \frac{1}{4}\right] a_s + a_{s-2}\right\} x^{s+\rho} = 0$$

$$\left(\rho^2 - \frac{1}{4}\right) a_0 x^\rho + \left[(1+\rho)^2 - \frac{1}{4}\right] a_1 x^{\rho+1} + \sum \left\{\left[(s+\rho)^2 - \frac{1}{4}\right] a_s + a_{s-2}\right\} x^{s+\rho} = 0$$

$$\left(\rho^2 - \frac{1}{4}\right) a_0 = 0 \Leftrightarrow a_0 = 0 \lor \left(\rho^2 - \frac{1}{4}\right) = 0 \Leftrightarrow a_0 = 0 \lor \boxed{\rho = \pm\frac{1}{2}}$$

$$\left[(1+\rho)^2 - \frac{1}{4}\right] a_1 = 0 \Leftrightarrow a_1 = 0 \lor \left(1 \pm 1 + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}}\right) = 0$$

$$a_s = -\frac{a_{s-2}}{(s+\rho)^2 - \frac{1}{4}} = -\frac{a_{s-2}}{s^2 \pm s + \cancel{\left(\frac{1}{4} - \frac{1}{4}\right)}} = -\frac{a_{s-2}}{s(s \pm 1)}$$

$\rho = +\frac{1}{2} \Rightarrow a_1 = 0 \Rightarrow a_{2m-1} = 0$

$$a_{2m} = -\frac{a_{2m-2}}{2m(2m+1)} = \frac{(-1)^m a_0}{(2m+1)!}$$

$$y = \sqrt{x} \sum_{m=0}^{\infty} \frac{(-1)^m a_0 x^{2m}}{(2m+1)!} = \frac{a_0}{\sqrt{x}} \underbrace{\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}}_{\sin x}$$

$\rho = -\frac{1}{2} \Rightarrow a_1 \neq 0 \Rightarrow a_{2m+1} = -\dfrac{a_{2m-1}}{(2m+1)(2m-1)} = \dfrac{(-1)^m a_1}{(2m+1)!}$