Sada příkladů 2/5

Číselné řady

Číselné řady s nezápornými členy

1. Nalezněte n-tý částečný součet a součet řady
$$\sum_{n=1}^{\infty} n^2 .$$

2. Spočtěte
$$\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{n}{2}]}}{2^n} .$$

3. Spočtěte
$$\sum_{n=1}^{\infty} (a+nd)q^n , \quad a, d \in \mathbb{R} , \quad |q| < 1 .$$

Sečtěte

4. $$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} .$$

5. $$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} .$$

6. Na základě elementárních úvah rozhodněte zda řady konvergují či divergují
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$
$$\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$$
$$\sum_{n=1}^{\infty} \operatorname{tg} \frac{\pi}{4n} .$$

Použitím kritérií pro konvergenci řad s nezápornými členy rozhodněte o konvergenci či divergenci následujících řad. Pokud řada obsahuje parametry, proveďte vzhledem k nim diskusi

7.

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right)$$

8.

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$$

9.

$$\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$$

10.

$$\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$$

11.

$$\sum_{n=1}^{\infty} \frac{\ln(n!)}{(n)^{\alpha}}, \quad \alpha \in \mathbb{R}$$

12.

$$\sum_{n=1}^{\infty} (n^{n^{\alpha}} - 1), \quad \alpha \in \mathbb{R}$$

13.

$$\sum_{n=1}^{\infty} (n^{\frac{1}{n^2+1}} - 1)$$

14.

$$\sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{1 \cdot 5 \cdot 9 \ldots (4n-3)}$$

15.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$$

16.

$$\sum_{n=1}^{\infty} \frac{n^2}{(\frac{\pi}{3} + \frac{1}{n})^n}$$

17.

$$\sum_{n=1}^{\infty} \frac{n^n}{(2n^2+n+1)^{\frac{n}{2}}}$$

18.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad p \in \mathbb{R}$$

19.

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p(\ln\ln n)^q}, \quad p, q \in \mathbb{R}$$

20.

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\sqrt[3]{n}}$$

21.

$$\sum_{n=1}^{\infty} \frac{p(p+1)\cdots(p+n-1)}{n!} \frac{1}{n^q}, \quad p, q \in \mathbb{R}$$

22.

$$\sum_{n=1}^{\infty} \Big(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\Big)^p, \quad p \in \mathbb{R}.$$

Číselné řady

$\{a_k\} \subset \mathbb{R}$ posloupnost $\rightarrow$ $\sum_{k=1}^{\infty} a_k$ řada $s_n = \sum_{k=1}^{n} a_n$ n-tý částečný součet

If $\exists$ $s := \lim_{n\to\infty} s_n$ ... konverguje (s vlastní) nebo diverguje (s nevlastní)

If $\neg\exists$ $\rightarrow$ osciluje $\rightarrow$ $s = \sum_{k=1}^{\infty} a_k$

- geometrické ... $a_k = a_0 q^k$
- harmonické ... $\sum_{k=1}^{\infty} \frac{1}{k}$
- teleskopické ... $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum \left(\frac{1}{k} - \frac{1}{k-1}\right) = \dots$

Nutná podmínka konvergence: $\lim_{k\to\infty} a_k = 0$

B-C podmínka: $\sum_{k=1}^{\infty} a_k$ Ⓚ $\iff$ splňuje B-C podmínku: $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N} \cap [n_0, \infty)$
$\forall p \in \mathbb{N}: \sum_{k=n+1}^{n+p} a_k < \varepsilon.$

- aritmetické: nemůže Ⓚ

Aritmetika řad $\sum(\alpha a_k + \beta b_k) = \alpha A + \beta B$

Absolutní konvergence, neabsolutní konvergence

(AK) $\Longrightarrow$ Ⓚ

Srovnávací kritérium I: $b_k \geq 0$, $|a_k| \leq b_k$ $\sum b_k$ Ⓚ $\rightarrow$ $\sum a_k$ (AK)

Leibnizovo kritérium: $\{a_k\}$ nezáporné nerostoucí $\Rightarrow$ $\sum (-1)^k a_k$ Ⓚ $\iff \lim_{k\to\infty} a_k = 0$

Řady s nezápornými členy

Srovnávací krit. II $\{a_k\}, \{b_k\} \subset [0,\infty)$, $k_0 \in \mathbb{N}$ a je splněna alespoň 1 z podmínek:

i) $a_k \geq b_k$ $\forall k \geq k_0$
ii) $\frac{a_{k+1}}{a_k} \geq \frac{b_{k+1}}{b_k}$ $\forall k \geq k_0$ a $\sum a_k$ Ⓚ $\Rightarrow$ $\sum b_k$ Ⓚ

Limitní srovnávací krit. $\{a_k\}, \{b_k\} \subset (0,\infty)$ a $\lim_{k\to\infty} \frac{a_k}{b_k} \in (0,\infty)$

$\Rightarrow$ $\sum a_k$ Ⓚ $\iff$ $\sum b_k$ Ⓚ

if $\lim \frac{a_k}{b_k} \in [0,\infty)$ a $\sum b_k$ Ⓚ $\Rightarrow$ $\sum a_k$ Ⓚ

Integrální kritérium: $a \in \mathbb{N}$, $f: \mathbb{R} \to \mathbb{R}$ spojitá kladná nerostoucí na $[a,\infty)$

Pak $\sum f(k)$ Ⓚ $\iff$ $(N)\int_a^{\infty} f\,dx \in \mathbb{R}$

$\rightarrow$ $\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ Ⓚ $\iff$ $\alpha > 1$ $\sum_{k=2}^{\infty} \frac{1}{k^\alpha \ln^\beta k}$ Ⓚ $\iff$ $\alpha > 1 \vee (\alpha = 1\ \&\ \beta > 1)$

Cauchyovo odmocninové krit.: $\{a_k\} \subset [0,\infty)$ $k_0 \in \mathbb{N}$

(i) $\exists q \in [0,1)$, že $\sqrt[k]{a_k} \le q \ \forall k \ge k_0 \Rightarrow \sum a_k$ Ⓚ

Spec. $\lim_{k\to\infty} \sqrt[k]{a_k} < 1 \Rightarrow \sum a_k$ Ⓚ

(ii) $\sqrt[k]{a_k} \ge 1 \ \forall k \ge k_0 \Rightarrow \sum a_k$ Ⓓ

Spec. $\lim_{k\to\infty} \sqrt[k]{a_k} > 1 \Rightarrow \sum a_k$ Ⓓ

d'Alembertovo podílové krit.: $\{a_k\} \subset (0,\infty)$ $k_0 \in \mathbb{N}$

(i) $\exists q \in [0,1)$, že $\frac{a_{k+1}}{a_k} \le q \ \forall k \ge k_0 \Rightarrow \sum a_k$ Ⓚ Spec. $\lim \frac{a_{k+1}}{a_k} < 1 \Rightarrow \sum a_k$ Ⓚ

(ii) $\frac{a_{k+1}}{a_k} \ge 1 \ \forall k \ge k_0 \Rightarrow \sum a_k$ Ⓓ. Spec. $\lim \frac{a_{k+1}}{a_k} > 1 \Rightarrow \sum a_k$ Ⓓ

Raabeho kritérium: $\{a_k\} \subset (0,\infty)$ $k_0 \in \mathbb{N}$

(i) $\exists q > 1$, že $k\left(\frac{a_k}{a_{k+1}} - 1\right) \ge q \ \forall k \ge k_0 \Rightarrow \sum_{k=1}^{\infty} a_k$ Ⓚ $\Leftarrow \lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) > 1$

(ii) $k\left(\frac{a_k}{a_{k+1}} - 1\right) \le 1 \ \forall k \ge k_0 \Rightarrow \sum a_k$ Ⓓ $\Leftarrow \lim_{k\to\infty} k\left(\frac{a_k}{a_{k+1}} - 1\right) < 1$

Gaussovo kritérium: $\{a_k\} \subset (0,\infty)$ $\exists p,q \in \mathbb{R}$ a $\varepsilon, C > 0$ že

$$\frac{a_k}{a_{k+1}} = p + \frac{q}{k} + \frac{t_k}{k^{1+\varepsilon}} \quad \text{kde } |t_k| \le C$$

(i) $p > 1 \Rightarrow \sum a_k$ Ⓚ $\quad p < 1 \Rightarrow \sum a_k$ Ⓓ

(ii) $p = 1$ a $q > 1 \Rightarrow \sum a_k$ Ⓚ

(iii) $p = 1$ a $q \le 1 \Rightarrow \sum a_k$ Ⓓ

Lobačevského kondenzační kritérium $\{a_k\} \subset [0,\infty)$ nerostoucí posloupnost.

Pak $\sum a_k$ Ⓚ $\Leftrightarrow \sum 2^k a_{2^k}$ Ⓚ

Pokrač. = Sada příkladů 2/6

P6-1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

1. Nalezněte s_n a s: $\sum_{k=1}^{\infty} k^2$

$$s_n = \frac{n(n+1)(2n+1)}{6}$$ … indukcí

$$\sum_{k=1}^{n}\left[(1+k)^3 - k^3\right] \qquad s_n = -1^3 + \cancel{2^3 - 2^3} + 3^3 + \dots \cancel{+k^3} + (n+1)^3 = (n+1)^3 - 1$$

$$\sum_{k=1}^{n}\left[1 + 3k + 3k^2 + \cancel{k^3} - \cancel{k^3}\right] = s_n = n + \frac{3}{2}n(n+1) + 3\sum_{k=1}^{n} k^2$$

$$\to \sum_{k=1}^{n} k^2 = \frac{(n+1)^3 - n - 1}{3} - \frac{n(n+1)}{2} = \frac{2(n+1)((n+1)^2-1) - 3n(n+1)}{6}$$

$$= \frac{(n+1)\left[2(n+1)^2 - 3n - 2\right]}{6} = \frac{n+1}{6}\underbrace{(2n^2 + 4n + \cancel{2} - 3n - \cancel{2})}_{2n^2+n} = \frac{n(n+1)(2n+1)}{6} \checkmark$$

$$= n(2n+1)$$

$$s_n \xrightarrow{n\to\infty} \infty = s \quad Ⓓ$$

2. $\sum_{k=1}^{\infty} k^3$ $\qquad \sum\left[(1+k)^4 - k^4\right] =$

$$s_n = -1^4 \dots + (n+1)^4 = (n+1)^4 - 1$$

$$s_n = \sum\left(1 + 4k + 6k^2 + \underline{4k^3} + \cancel{k^4} - \cancel{k^4}\right)$$

$$\boxed{4\sum_{1}^{n} k^3} = (n+1)^4 - 1 - n - 4\frac{n(n+1)}{2} - \cancel{6}\frac{n(n+1)(2n+1)}{\cancel{6}}$$

$$= (n+1)^4 - 1 - n - 2n(n+1) - n(n+1)(2n+1)$$

$$= 4(n+1)^4 - 4 - n - n(n+1)(2n-1)$$

$$= \underline{n^4} + \underline{4n^3} + \underline{6n^2} + \cancel{4n} + \cancel{1} - \cancel{1} - \cancel{n} \; \underline{-2n^2} - \cancel{2n} \; \underline{-2n^3} \; \underline{-3n^2} - \cancel{n} = n^4 + 2n^3 + n^2 = \boxed{n^2(n+1)^2}$$

$$\sum\left[k^4 - (k-1)^4\right] = -0^4 + 1^4 - 1^4 + 2^4 \dots - (n-1)^4 + n^4 = n^4$$

$$\cancel{k^4 - 4k^3 + 6k^2 - 4k + 1}$$

$$\sum\left[\cancel{k^4} - \cancel{k^4} + 4k^3 - 6k^2 + 4k - 1\right] = 4\sum k^3 - 6\sum k^2 + 4\sum k - \sum 1$$

$$n^4 = \underline{4\sum k^3} - n(n+1)(2n+1) + 2n(n+1) - n$$

$$4\sum k^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$= n^4 + 2n^3 + 3n^2 + \cancel{n} - 2n^2 - \cancel{2n} + \cancel{n}$$

$$= n^4 + 2n^3 + n^2$$

$$= n^2(n^2 + 2n + 1)$$

$$= n^2(n+1)^2$$

$$\sum_{1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{1}^{n} k\right)^2$$

86-2

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \left[\frac{n}{2}\right] = & 0 & 1 & 1 & 2 & 2 & 3 & ? \end{array}$$

celá část $\frac{n}{2}$

2. $\sum_{1}^{\infty} \frac{(-1)^{\left[\frac{n}{2}\right]}}{2^n} = \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} -$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2^{4k}} + \sum_{k=1}^{\infty} \frac{1}{2^{4k+1}} - \sum \frac{1}{2^{4k-2}} - \sum \frac{1}{2^{4k-1}}$$

$$= \underbrace{\sum_{k=1}^{\infty} \frac{1}{2^{4k}}}_{\text{geom}} - \sum \frac{1}{2^{4k-1}} - \sum \frac{1}{2^{4k-2}} + \sum_{1} \frac{1}{2^{4k-3}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^{4k}} - 2 \sum_{1} \frac{1}{2^{4k}} - 4 \sum_{1} \frac{1}{2^{4k}} + 8 \sum_{1} \frac{1}{2^{4k}}$$

$$= 3 \sum_{k=1}^{\infty} \frac{1}{2^{4k}} = 3 \left(\underbrace{\sum_{0}^{\infty} \frac{1}{(2^4)^k}} - \frac{1}{2^0} \right)$$

$$\frac{1}{1-q} = \frac{1}{1-\frac{1}{2^4}} = \frac{1}{1-\frac{1}{16}} = \frac{16}{15}$$

$$= 3\left(\frac{16}{15} - 1\right) = 3 \cdot \frac{1}{15} = \frac{1}{5}$$

Or: $3 \cdot \frac{1}{2^4} \frac{1}{1-\frac{1}{2^4}} = \frac{3}{16} \cdot \frac{1}{1-\frac{1}{16}} = \frac{3}{\cancel{16}} \frac{\cancel{16}}{15} = \frac{1}{5}$

3. $\sum_{1}^{\infty} (a + nd) q^n \quad a, d \in \mathbb{R} \quad |q| < 1$

$$= \sum_{1}^{\infty} a q^n + \sum_{1}^{\infty} d n q^n = ~~aq \sum_{1}^{\infty} q^{n-1}~~$$

$$= aq \sum_{1}^{\infty} q^{n-1} = aq \sum_{0}^{\infty} q^n = aq \frac{1}{1-q} = \frac{aq}{1-q}$$

$$\sum_{1}^{\infty} d n q^n = d\left(q + 2q^2 + 3q^3 + \dots\right)$$

$$\begin{array}{llll} q & + q^2 & + q^3 & \\ & q^2 & + q^3 & + \\ & & q^3 & + \dots \end{array}$$

$q \times \frac{1}{1-q}$

$q^2 \times \frac{1}{1-q}$

$q^3 \times \frac{1}{1-q}$

$$\frac{1}{1-q} \underbrace{\left(q + q^2 + q^3 + \dots\right)}_{q \cdot \frac{1}{1-q}} = \frac{q}{(1-q)^2}$$

$$= \frac{aq}{1-q} + \frac{dq}{(1-q)^2} = \frac{q}{1-q}\left(a + \frac{d}{1-q}\right)$$

P 6-3

4. $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3} \right)$

$s_n = \frac{1}{3}\left[\underline{\frac{1}{1}} - \frac{1}{4} + \underline{\frac{1}{2}} - \frac{1}{5} + \underline{\frac{1}{3}} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} \dots - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} \right]$

$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} \right) = \frac{1}{3} \frac{6+3+2}{6} = \underline{\frac{11}{18}}$

5. $\sum_{1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$ $\quad \frac{2n+1}{n^2(n+1)^2} = \frac{A}{n+1} + \frac{B}{(n+1)^2} + \frac{C}{n} + \frac{D}{n^2} = \underbrace{\frac{1}{n^2} - \frac{1}{(n+1)^2}}_{n^2+2n+1-n^2}$

$\rightsquigarrow A=0,\ C=0,\ B=-1,\ D=1$

$= \underbrace{\sum_{1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)}_{= \frac{(n+1)^2 - n^2}{n^2(n+1)^2}} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{3^2} \dots$

$= 1$

6. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\quad a_n = \frac{1}{n^2+1} < \frac{1}{n^2} \underset{(n>2)}{<} \frac{1}{n(n-1)}$ konverguje

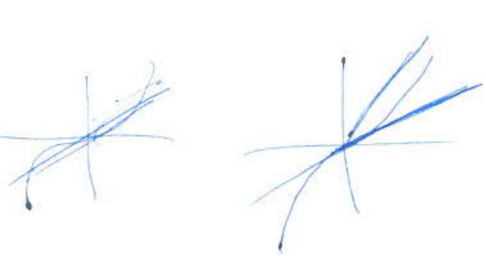
↳ konverguje.

$(k+1)^2 = k^2+2k+1 > k(k+2) = k^2+2k$

$\sum_{1}^{\infty} \frac{k+1}{k(k+2)}$ $\quad a_k = \frac{k+1}{k(k+2)} > \frac{k+1}{(k+1)^2} = \frac{1}{k+1}$ $\quad \sum_{k=1}^{\infty} \frac{1}{k+1} = \sum_{k=2}^{\infty} \frac{1}{k}$ $\underline{D}$

$a_{k-1} = \frac{k}{(k-1)(k+1)} > \frac{k}{k^2} = \frac{1}{k}$ Diverguje

$\sum_{1}^{\infty} \operatorname{tg} \frac{\pi}{4n}$



$\operatorname{tg} \frac{\pi}{4n} \geq \frac{\pi}{4n}$ diverguje $\rightarrow \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{1}{n}$ $\underline{D}$

(P6-4)

7. $\sum_{1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n-1}\right) = \sum \frac{1}{n} \frac{n+1-(n-1)}{\sqrt{n+1} + \sqrt{n-1}} = \sum \frac{2}{n\left(\sqrt{n+1} + \sqrt{n-1}\right)} \le$

$(\sqrt{n+1} + \sqrt{n-1} \geq \sqrt{n})$

$\le \sum \frac{2}{n\sqrt{n}} = 2 \sum \frac{1}{n^{3/2}}$ ✓

$\sqrt{n} < \sqrt{n+1} + \sqrt{n-1}$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sim n^{-3/2}$ ✓

[See Vyšíral - 2 - 7g]

Ⓚ

~~8. $\sum_{2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ → See Vyšíral 8-4d~~

~~9. $\sum_{3} \frac{1}{(\ln n)^{\ln \ln n}}$ Comparison test → diverges.~~

~~10. $\sum_{1}^{\infty} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$ Converges when $\frac{\ln(n!)}{n^{\alpha}} \to 0$~~

~~11. $\sum \frac{\ln(n!)}{n^{\alpha}}$ $\alpha \in \mathbb{R}$~~

8. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

(Same as Vyšíral - 2 - 4d)

$(\ln n)^{\ln n} = e^{\ln n (\ln \ln n)} \underset{\uparrow}{>} e^{2 \ln n} = e^{\ln n^2} = n^2$

for $\ln \ln n > 2$

$\ln n > e^2$

$n > e^{e^2} = 1618.2$

$\sum_{n=1615}^{\infty} \frac{1}{(\ln n)^{\ln n}} \le \sum_{n=1615}^{\infty} \frac{1}{n^2}$ ✓ Ⓚ

9. $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$ $(\ln n)^{\ln \ln n} = e^{(\ln \ln n)^2} < e^{\left(\sqrt{\ln n}\right)^2} = e^{\ln n} = n$

$\ln x \underset{\substack{\uparrow \\ x > x_0}}{<} \sqrt{x}$

$\sum \frac{1}{(\ln n)^{\ln \ln n}} > \sum \frac{1}{n}$ X Ⓓ

10. $\sum_{1}^{\infty} \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$ ... $a_n = \frac{\cancel{n^n} \sqrt[n]{n}}{\cancel{n^n}\left(1+\frac{1}{n^2}\right)^n} \to 1$ X

Ⓓ

11. $\sum_{n=1}^{\infty} \frac{\ln(n!)}{n^{\alpha}}$ $x \in \mathbb{R}$

$(n-1)\ln 2 < \ln n! < (n-1)\ln n$

$\ln n! = \ln 1 + \ln 2 + \dots + \ln n < (n-1)\ln 2$ $\sum_{1}^{\infty} \frac{\ln n}{n^{\alpha-1}}$

$a_n < \ln n \frac{n-1}{n^{\alpha}} < \ln n \frac{1}{n^{\alpha-1}}$ ... konverguje pro $\alpha > 2$ Ⓚ

$\int_{1}^{\infty} \frac{\ln x}{x^{\alpha-1}}\, dx$ Ⓓ pro $\alpha \leq 2$

$= \dots \left[\frac{1}{x^{\alpha}}\right]_1^{\infty} - \frac{1}{(2-\alpha)^2}\left[x^{2-\alpha}\right]_1^{\infty}$ K pro $\alpha \geq 2$

- 12. $\sum_{n=1}^{\infty}\left(n^{n^{\alpha}} - 1\right)$ $\alpha \in \mathbb{R}$ $\boxed{\text{KONDENZAČNÍ ?}}$

$a_n = e^{n^{\alpha}\ln n} - 1 = \underbrace{\frac{e^{n^{\alpha}\ln n} - 1}{n^{\alpha}\ln n}}_{\lim\limits_{x\to 0}\frac{e^{x}-1}{x} = 1} \underbrace{n^{\alpha}\ln n} \overset{!}{\to} 0$ ✓ Nutná podmínka.

$\lim\limits_{n\to\infty} \frac{\ln n}{n^{-\alpha}} \overset{l'H}{=} \lim \frac{1}{\cancel{n}} \cancel{n^{\alpha}x}$

$\beta > 0$

$\to 1$ if $n^{\alpha}\ln n \to 0$ ... $\alpha < 0$

Now: limitní ~~[illegible]~~ srovnávací kritérium: $\frac{a_n}{b_n} = \frac{a^{n^{\alpha}\ln n} - 1}{n^{\alpha}\ln n} \xrightarrow[\alpha<0]{} 1$ ... $\sum a_n$ K

$\Updownarrow$

$\sum b_n$ K

$\Updownarrow$

$\alpha < -1$

- 13. $\sum_{1}^{\infty}\left(n^{\frac{1}{n^2+1}} - 1\right)$

$a_n = n^{\frac{1}{n^2+1}} - 1 < n^{\frac{1}{n^2}} - 1 = \sqrt[n^2]{n} - 1$

$a_n = \underbrace{\frac{e^{\frac{\ln n}{n^2+1}} - 1}{\frac{\ln n}{n^2+1}}}_{\to 1} \frac{\ln n}{n^2+1} \overset{!}{\Longrightarrow} 0$ ✓ Nutná podmínka,

$\frac{\ln n}{n^2+1} \to 0$ $\frac{\ln n}{n^2+1} \to 0$

+ Limitní srovnávací kritérium

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{e^{\frac{\ln n}{n^2+1}} - 1}{\frac{\ln n}{n^2+1}} = 1 \longrightarrow \sum\left(n^{\frac{1}{n^2+1}} - 1\right)$ Ⓚ

$\sum \frac{\ln n}{n^2+1} < \sum \frac{\ln n}{n^2}$ ... K

$\sum b_n = \sum \frac{\ln n}{n^{\beta}}$ $\frac{a_n}{b_n} = \frac{\frac{\ln n}{n^{\beta}}}{\frac{1}{n^{\tilde{\beta}}}} = \frac{\ln n}{n^{\beta-\tilde{\beta}}} = \frac{\frac{1}{n}}{(\beta-\tilde{\beta})\, n^{\beta-\tilde{\beta}-1}} := \frac{1}{n^{\beta-\tilde{\beta}}} \to 0$

$\beta > 1$ $\beta > \tilde{\beta} > 1$

14. $$\sum_{1}^{\infty} \frac{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-1)}{1\cdot 5\cdot 9\cdot\ldots\cdot(4n-3)}$$

[illegible] ~~podílové~~. Limitní podílové krit.:

$$\frac{a_{n+1}}{a_n} = \underbrace{\frac{\boxed{2}\,\cancel{5\cdot 8\cdot 11\cdot\ldots\cdot(3n-1)}(3n+2)}{\boxed{1}\,\cancel{5\cdot 9\cdot 13\cdot\ldots\cdot(4n-3)}(4n+1)}}_{a_{n+1}} \cdot \underbrace{\frac{\boxed{1}\,\cancel{5\cdot 9\cdot\ldots\cdot(4n-3)}}{\boxed{2}\,\cancel{5\cdot 8\cdot\ldots\cdot(3n-1)}}}_{\frac{1}{a_n}} = \frac{3n+2}{4n+3} \xrightarrow{n\to\infty} \frac{3}{4} < 1 \quad ✓$$

$n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = \int_0^\infty \exp(n\ln x - x)\,dx$ … Taylor okolo $x = n$

$$\hookrightarrow n! \approx \int_0^\infty \exp\left(n\ln n - n - \frac{(x-n)^2}{2n} + o((x-n)^2)\right)dx \approx \left(\frac{n}{e}\right)^n \int_0^\infty -\frac{(x-n)^2}{2n}\,dx$$
$$= \sqrt{2n}\left(\frac{n}{e}\right)^n \int_{-\sqrt{\frac{n}{2}}}^{\infty} e^{-x^2}dx \approx \sqrt{2n}\left(\frac{n}{e}\right)^n \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

15. $$\sum_{1}^{\infty} \frac{(n!)^2}{2^{n^2}}$$

$$a_n = \frac{(n!)^2}{2^{n^2}} \longrightarrow \frac{2\pi n\left(\frac{n}{e}\right)^{2n}}{2^{n^2}} = ~~2\pi n\left(\frac{n}{2e}\right)~~\ 2\pi\,\frac{n^{2n+1}}{2^{n^2}e^{2n}} \ \ldots$$

Stirling: $n! \rightsquigarrow \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{2^{(n+1)^2}}\,\frac{2^{n^2}}{(n!)^2} = (n+1)^2\,\frac{2^{n^2}}{2^{n^2+2n+1}} = \frac{(n+1)^2}{2^{2n+1}} = \frac{n^2+2n+1}{2^{2n+1}}$$

$$\approx \frac{n^2}{2^{2n+1}} = \frac{e^{2\ln n}}{e^{(2n+1)\ln 2}} = e^{2\ln n - (2n+1)\ln 2}$$

$\lim 2\ln n$

$\sim \frac{n^2}{4^n} \to 0$ Ⓚ dle limitního podílového.

16. $$\sum_{n=1}^{\infty} \frac{n^2}{\left(\frac{\pi}{3}+\frac{1}{n}\right)^n} \qquad \sqrt[n]{a_n} = \frac{n^{2/n}}{\frac{\pi}{3}+\frac{1}{n}} \longrightarrow \frac{1}{\frac{\pi}{3}} = \frac{3}{\pi} < 1$$

Ⓚ dle limitního odmocninového

17. $\sum_{1}^{\infty} \frac{n^n}{(2n^2+n+1)^{n/2}}$ $\quad \sqrt[n]{a_n} = \frac{n}{\sqrt{2n^2+n+1}} = \frac{\cancel{n}}{\cancel{n}\sqrt{2+\frac{1}{n}+\frac{1}{n^2}}} \to \frac{1}{\sqrt{2}} < 1$

Ⓚ dle limitního odmocninového

18. $\sum_{2}^{\infty} \frac{1}{n(\ln n)^p} \quad p \in \mathbb{R}$

Kondenzační krit.: $\sum a_n \; \underline{K} \iff \sum 2^n a_{2^n} \; K$

$$\to \sum \cancel{2^n} \frac{1}{\cancel{2^n}(\ln 2^n)^p} = \sum \frac{1}{(n \ln 2)^p} = \frac{1}{(\ln 2)^p} \underbrace{\sum \frac{1}{n^p}}$$

Ⓚ for $p>1$

Integrální: $\int_{2}^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} \frac{dt}{t^p} = \left[\frac{t^{-p+1}}{-p+1} \right]_{\ln 2}^{\infty}$

$\uparrow$ $p>0$ ($f(x)$ nerostoucí)

$\ln x = t$

$\frac{dx}{x} = dt$

$t = \ln 2 \ldots \infty$

(... =

$p>1 \quad \frac{1}{-(p-1)} \frac{1}{t^{p-1}} \Big|_{\ln 2}^{\infty} = \frac{1}{p-1} \frac{1}{(\ln 2)^{p-1}}$ ✓

$p=1 \quad \ln t \Big|_{\ln 2}^{\infty} \to \infty$

$p<1 \quad \frac{t^{1-p}}{1-p} \Big|_{\ln 2}^{\infty} \to \infty$

19. $\sum_{3}^{\infty} \frac{1}{n(\ln n)^p(\ln\ln n)^q}$ K ~ Př Q

Case $q = 0$: Příklad 18. K pro $p > 1$

Case $q > 0$:

- If $p > 1$ ✓ $\frac{1}{n(\ln n)^p(\ln\ln n)^q} < \frac{1}{n(\ln n)^p}$ ✓
- If $p = 1$ $\sum_{3}^{\infty} \frac{1}{n\ln n(\ln\ln n)^q}$

Integrální: $\int_{3}^{\infty} \frac{dx}{x\ln x(\ln\ln x)^q} \underset{\ln x = t}{=} \int_{\ln 3}^{\infty} \frac{dt}{t(\ln t)^q}$

$\ln x = t$

$\frac{dx}{x} = dt$

$t = \ln 3 \dots \infty$

$\underset{\ln t = u}{=} \int_{\ln\ln 3}^{\infty} \frac{du}{u^q}$ ... Konvrg pro $q > 1$

$\ln t = u$

$\frac{dt}{t} = du$

$u = \ln\ln 3 \dots \infty$

20. $\sum_{1}^{\infty} e^{-\sqrt[3]{n}}$ $=$

$$a_n = \frac{1}{e^{n^{1/3}}} \le \frac{1}{e^{\ln n^2}} = \frac{1}{n^2} \text{ ... konverge}$$

$$\sqrt[3]{n} \ge 2\ln n \quad \text{pro } n > n_0$$

$$= \ln n^2$$

$$\longrightarrow \quad n > 8\ln^3 n \quad \dots \quad n \gtrsim 4800$$

21. $\sum_{1}^{\infty} \frac{p(p+1)\dots(p+n-1)}{n!}\frac{1}{n^q}$ $\quad p,q \in \mathbb{R}$ $\longrightarrow$ $\boxed{\text{See next page}}$

$$\frac{a_{n+1}}{a_n} = \frac{p\cancel{(p+1)\dots(p+n-1)}(p+n)}{(n+1)!}\frac{1}{(n+1)^q} \times \frac{n!\,n^q}{\cancel{p(p+1)\dots(p+n-1)}}$$

$$= \cancel{\frac{n+p}{n+q}\left(\frac{n}{n+1}\right)^q} \quad \frac{p+n}{n+1}\left(\frac{n}{n+1}\right)^q$$

Raabe $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = q$ $\begin{cases} q > 1 \dots \underline{K} \\ q < 1 \dots \underline{D} \end{cases}$

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = n\,\frac{n+1}{p+n}\left(\frac{n+1}{n}\right)^q = \frac{(n+1)^2}{n+p}\left(\frac{n+1}{n}\right)^{q-1}$$

22. $\sum_{1}^{\infty}\left(\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\right)^p$ $\quad p \in \mathbb{R}$ $\longrightarrow$ $\boxed{\text{See next page + 1}}$

$$\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+2}\right)^p = \left(1 - \frac{1}{2n+2}\right)^p = 1 - \frac{p}{2n+2} + o\left(\frac{1}{n^2}\right)$$

$$= 1 - \frac{p}{2}\frac{1}{n} + o\left(\frac{1}{n^2}\right)$$

㉑ $\sum_{n=1}^{\infty} \frac{p(p+1)\dots(p+n-1)}{n!}\frac{1}{n^q} \qquad p,q \in \mathbb{R}$

- Podílové? $\quad \frac{a_{n+1}}{a_n} = \frac{p+n}{n+1}\left(\frac{n}{n+1}\right)^q \xrightarrow{n\to\infty} 1 \quad \forall p,q \in \mathbb{R} \quad ↯?$

- Raabe? $\quad \frac{a_n}{a_{n+1}} = \frac{(n+1)^{q+1}}{(p+n)n^q} = 1 + \frac{(n+1)^{q+1}-(p+n)n^q}{(p+n)n^q}$

Generalized binomial theorem (see next page)

$$(n+1)^{q+1} = n^{q+1} + (q+1)n^q \cdot 1^1 + \frac{(q+1)q\,n^{q-1}}{2!} + \dots n^{q-2} \dots$$

$$(n+1)^{q+1} - (p+n)n^q = \cancel{n^{q+1}} + (q+1)n^q + \dots n^{q-1} + \dots - \cancel{n^{q+1}} - pn^q$$

$$= (q-p+1)n^q + \frac{q(q+1)}{2}n^{q-1} + \dots n^{q-2} + \dots$$

$$\frac{a_n}{a_{n+1}} = 1 + \frac{q-p+1}{p+n} + \frac{q(q+1)}{2(p+n)n} + \mathcal{O}\left(\frac{1}{n^3}\right)$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim\left[(q-p+1)\frac{n}{n+p} + \mathcal{O}\left(\frac{1}{n}\right)\right] = \underline{q-p+1}$$

$q-p+1 > \cancel{1} \iff q>p \Rightarrow K.$

$q-p+1 \le 1 \iff q \le p \Rightarrow D.$

- Pro $p=q$? $\quad \frac{a_n}{a_{n+1}} = 1 + \frac{1}{p+n} + \frac{p(p+1)}{4pn} + \mathcal{O}\left(\frac{1}{n^3}\right)$

$$= 1 + \frac{1}{n+p} + \frac{p+1}{4n} + \mathcal{O}\left(\frac{1}{n^3}\right)$$

$$\frac{1}{n} - \frac{p}{n^2} + \frac{p^2}{n^3} - \frac{p^3}{n^4} + \dots \quad \text{Taylor v } n=\infty$$

$$\to \frac{a_n}{a_{n+1}} = 1 + \underbrace{\left(1+\frac{p+1}{4}\right)\frac{1}{n}}_{\underbrace{\frac{p+5}{4}\frac{1}{n}}} + \dots$$

1

Ⓚ pro $\frac{p+4}{4} > 1 \iff p=q>0 \qquad$ Ⓓ pro $p=q \le 0$

㉒ $\sum_{1}^{\infty}\left(\frac{1\cdot 3\cdot\ \dots\ \cdot(2n-1)}{2\cdot 4\cdot\ \dots\ 2n}\right)^p$

- Podílové? $\frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+2}\right)^p \xrightarrow{n\to\infty} 1 \quad \forall p\in\mathbb{R}$ $\underline{\lightning ?}$

- Raabe? $\frac{a_n}{a_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \left(1+\frac{1}{2n+1}\right)^p$

generalized binomial

$$= 1 + \frac{p}{2n+1} + O\left(\frac{1}{n^2}\right)$$

$$(x+y)^p = \sum_{k=0}^{\boxed{\infty}} \binom{p}{k} x^{p-k} y^k = x^p + p x^{p-1} y + \frac{p(p-1)}{2!} x^{p-2} y^2 + \dots$$

$$x^p\left(1+\left(\frac{y}{x}\right)\right)^p \qquad \binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!} \qquad p\in\mathbb{R}$$

Taylor bodem $\left(\frac{y}{x}\right)=0$

Raabe: $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty}\left(\frac{np}{2n+1}+O\left(\frac{1}{n}\right)\right) = \boxed{\frac{p}{2}}$

$\frac{p}{2} > 1 \iff p > 2 \Rightarrow \sum a_n$ K.

$\frac{p}{2} \leq 1 \iff p \leq 2 \Rightarrow \sum a_n$ D.

Pro $p=2$? $\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{2n+1}\right)^2 = \left(1+\frac{1}{2n}+\frac{r_k}{n^2}\right)^2$

r_k omezený

$1+\frac{1}{2n+1}$ Taylor v $n=\infty$ + zbytek Taylorova rozvoje

$$= 1 + \frac{\boxed{1}}{n} + \underbrace{\frac{2r_k+\frac{1}{4}}{n^2} + \frac{r_k}{n^3} + \frac{r_k^2}{n^4}}_{\frac{t_k}{n^2}}$$

$t_k = 2r_k + \frac{1}{4} + \frac{r_k}{k} + \frac{r_k^2}{k^2}$

D (Gauss: need $h > 1$ for K; ≤ 1 Ⓓ)