Sada příkladů 2/4

Obyčejné diferenciální rovnice

Lineární rovnice s konstantními koeficienty

Nalezněte obecná řešení rovnic

1.
$$y^{III} - 3y'' + 3y' - y = 0$$
$\lambda \in \mathbb{R}$ 3-násobný

2.
$$y'' - 2y' - 3y = \mathrm{e}^{4x}$$
$\lambda \in \mathbb{R}$ + RHS

3.
$$y'' - y = 2\mathrm{e}^{x} - x^2$$
$\lambda \in \mathbb{R}$ + 2 × RHS

4.
$$y'' - 3y' + 2y = \sin x$$
$\lambda \in \mathbb{R}$ + RHS

5.
$$y'' + 4y' - 5y = 2\mathrm{e}^{x}\sin^2 x$$
$\lambda \in \mathbb{R}$ + RHS

6.
$$y'' - 2y' + y = 2x\mathrm{e}^{x} + \mathrm{e}^{x}\sin 2x$$
$\lambda \in \mathbb{R}$ 2-násobný + RHS

7.
$$y^{IV} - 5y'' + 4y = \sin x \cos 2x$$
$\lambda \in \mathbb{R}$ + RHS

lze spec. RHS

8.
$$y'' - 2y' + y = \frac{\mathrm{e}^{x}}{x}$$
$\lambda \in \mathbb{R}$ 2násobný

9.
$$y'' + 4y = 2\mathrm{tg}\, x$$
λ imaginární

variace konstant

10.
$$x^2 y^{III} = 2y'$$
Euler

11.
$$x^2 y'' + xy' + 4y = 10x.$$
Euler → modif. spec. RHS

Lineární rovnice n-tého řádu

Nalezněte obecná řešení rovnic, znáte-li jedno řešení homogenní rovnice

12.

$$(2x+1)y'' + 4xy' - 4y = 0, \quad y = \mathrm{e}^{ax}$$

13.

$$xy'' + 2y' - xy = 0, \quad y = \frac{\mathrm{e}^x}{x}$$

14.

$$(x+1)xy'' + (x+2)y' - y = x + \frac{1}{x}, \quad y = x + 2$$

15.

$$(2x+1)y'' + (2x-1)y' - 2y = x^2 + x.$$

Jedno řešení je ve tvaru polynomu.

Jiné typy ODR

16.

$$2yy' = y^2 + y'^2$$

17.

$$x^2y'' = y'^2$$

18.

$$y^3y'' = 1$$

19.

$$y'' = \mathrm{e}^y$$

20.

$$y'' + y'^2 = 2\mathrm{e}^{-y}.$$

① $y''' - 3y'' + 3y' - y = 0$ ODR 3. řádu konst. koef. RHS = 0

$\to e^{\lambda x}$

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0 \iff \lambda = 1$ trojnásobný kořen

$\to$ Fundamentální systém $\{e^{\lambda x}, x e^{\lambda x}, x^2 e^{\lambda x}\}$

$\to y = c_0 e^{\lambda x} + c_1 x e^{\lambda x} + c_2 x^2 e^{\lambda x}$ $c_0, c_1, c_2 \in \mathbb{R}$ $x \in \mathbb{R}$

② $y'' - 2y' - 3y = e^{4x}$

y_H: $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \to FS = \{e^{3x}, e^{-x}\} \to y_H = c_1 e^{3x} + c_2 e^{-x}$

y_P: variace konstant

spec. pravá strana: $f(x) = e^{4x}$, 4 není kořenem charakteristického polynomu

$\to y_P = a e^{4x}$

$f(x) = e^{\mu x}\left(P_1(x)\cos(\nu x) + P_2(x)\sin(\nu x)\right)$

$\mu = 4$ $\text{st} P_1 = 0$ $\nu = 0$

$\lambda = \mu + i\nu = 4$ není kořen char. pol.

$y' = 4a e^{4x}$

$y'' = 16 a e^{4x}$

ODR: $16 a \cancel{e^{4x}} - 2 \cdot 4 a \cancel{e^{4x}} - 3a \cancel{e^{4x}} = \cancel{e^{4x}}$

$5a = 1$

$a = \frac{1}{5} \to y_P = \frac{1}{5} e^{4x}$

$\underline{y = y_P + y_H = \frac{1}{5} e^{4x} + c_1 e^{3x} + c_2 e^{-x}}$ $x \in \mathbb{R}, c_1, c_2 \in \mathbb{R}$

$\to$ Variace konstant

$$\underbrace{\begin{pmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{pmatrix}}_{\text{Wronskiho matice}} \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{4x} \end{pmatrix} \leftarrow f(x)$$

vektor c_i'

$e^{-x} c_1' + e^{3x} c_2' = 0$

$-e^{-x} c_1' + 3e^{3x} c_2' = e^{4x}$

$4e^{3x} c_2' = e^{4x}$

$c_2' = \frac{e^x}{4}$

$c_2 = \frac{e^x}{4}$

$\to c_1' = -e^{4x} c_2' = -\frac{e^{5x}}{4}$

$c_1 = -\frac{e^{5x}}{20}$

$\to y_P = c_1(x) e^{-x} + c_2(x) e^{3x} = -\frac{1}{20} e^{4x} + \frac{e^{4x}}{4} = \left(-\frac{1}{20} + \frac{5}{20}\right) e^{4x} = \frac{e^{4x}}{5}$ ✓

$\to y = y_P + y_H = \frac{e^{4x}}{5} + c_1 e^{3x} + c_2 e^{-x}$

③ $y'' - y = 2e^x - x^2$

y_H: $\lambda^2 - 1 = (\lambda+1)(\lambda-1) = 0 \Leftrightarrow \lambda = \pm 1 \Rightarrow FS = \{e^x, e^{-x}\} \Rightarrow y_H = c_1 e^x + c_2 e^{-x}$

$$y_P = \underset{\substack{\uparrow \\ f_1(x) = 2e^x}}{y_{P1}} + \underset{\substack{\uparrow \\ f_2(x) = -x^2}}{y_{P2}}$$

y_{P1}: $f_1(x) = 2e^x$ ... $\lambda = +1$ je 1-násobný koren, st $P = 0$

y_{P2}: $f_2(x) = -x^2$ ... $\lambda = 0$ není koren (násobnost $= 0$), st $P = 2$

$\Rightarrow y_{P1} = Axe^x$

$y'_{P1} = Axe^x + Ae^x = Ae^x(1+x)$

$y''_{P1} = Ae^x(1+x) + Ae^x = Ae^x(2+x)$

ODR: $A\cancel{e^x}(2+x) - A\cancel{e^x}(1+x) = 2\cancel{e^x}$

$2A = 2$

$A = 1$ $\Rightarrow y_{P1} = xe^x$

$\Rightarrow y_{P2} = Bx^2 + Cx + D$

$y'_{P2} = 2Bx + C$

$y''_{P2} = 2B$

ODR: $2B - Bx^2 - Cx - D = -x^2$

x^2: $B = 1$

x^1: $C = 0$

x^0: $2B - D = 0 \Rightarrow D = 2$

$\Rightarrow y_{P2} = x^2 + 2$

$y = y_P + y_H = y_{P1} + y_{P2} + y_H = \underline{xe^x + x^2 + 2 + c_1 e^x + c_2 e^{-x}}$ $\quad x \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R}$

(4) $y'' - 3y' + 2y = \sin x$

y_H: $\lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0 \Leftrightarrow \lambda = 1 \vee \lambda = 2 \Rightarrow FS = \{e^x, e^{2x}\}$

$$y_H = c_1 e^x + c_2 e^{2x}$$

y_P: $f(x) = \sin x = e^{\overset{\mu}{0}x} \sin \overset{\nu}{1}x$

$\lambda = \mu + i\nu = 0 + 1\nu i = i\nu$ není kořen char. rce (násobnost = 0)

$\Rightarrow y_P = e^{0x}(A\cos 1x + B\sin 1x) = A\cos x + B\sin x$

$y_P' = -A\sin x + B\cos x$

$y_P'' = -A\cos x - B\sin x$

ODR: $-A\cos x - B\sin x + 3A\sin x - 3B\cos x + 2A\cos x + 2B\sin x = \sin x$

$\cos x$: $-A - 3B + 2A = 0 \rightarrow A - 3B = 0$

$\sin x$: $-B + 3A + 2B = 1 \qquad 3A + B = 1$

$10B = 1 \rightarrow B = \frac{1}{10} \qquad A = \frac{3}{10}$

$\Rightarrow y_P = \frac{1}{10}(3\cos x + \sin x)$

$y = y_P + y_H = \frac{1}{10}(3\cos x + \sin x) + c_1 e^x + c_2 e^{2x} \qquad x \in \mathbb{R} \quad c_1, c_2 \in \mathbb{R}$

(5) $y'' + 4y' - 5y = 2e^x \sin^2 x$

y_H: $\lambda^2 + 4\lambda - 5 = (\lambda+5)(\lambda-1) = 0 \Rightarrow \lambda_{1,2} = \begin{cases} 1 \\ -5 \end{cases} \Rightarrow y_H = c_1 e^x + c_2 e^{-5x}$

y_P: $f(x) = 2e^x \sin^2 x = 2e^x \dfrac{1-\cos(2x)}{2} = e^x(1-\cos(2x)) = e^x - e^x\cos(2x)$

$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \rightarrow \sin^2 x = \dfrac{1-\cos 2x}{2}$

y_{P1}: $f_1(x) = e^x$ ... $\lambda = 1 + 0i$ je kořen násob. 1 $\rightarrow y_{P1} = Axe^x$

y_{P2}: $f_2(x) = -e^x\cos(2x)$ ... $\lambda = 1 + 2i$ není kořen $\rightarrow y_{P2} = e^x(B\cos x + C\sin x)$

$\rightsquigarrow y = c_1 e^x + c_2 e^{-5x} + \frac{1}{6}xe^x + \frac{1}{40}e^x(\cos(2x) - 3\sin(2x))$ $\quad x \in \mathbb{R}$

$c_1, c_2 \in \mathbb{R}$

⑥ $y'' - 2y' + y = 2xe^x + e^x \sin(2x)$

y_H: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$ ... $\lambda = 1$ 2násobný koreň $\Rightarrow y_H = c_1 e^x + c_2 x e^x$

y_{P1}: $f_1(x) = 2xe^x$ ← $\lambda = 1$ je 2násobný koreň $\Rightarrow y_{P1} = x^2 e^x \underbrace{(A + Bx)}_{\text{st } P = 1}$, $A \to 0$, $B \to \frac{1}{3}$

st. $P = 1$; nás. koreň

y_{P2}: $f_2(x) = 1e^x \sin(2x)$ ... $\lambda = 1 + 2i$ nie je koreň $\Rightarrow y_{P2} = e^x (C\cos(2x) + D\sin(2x))$, $C \to 0$, $D \to -\frac{1}{4}$

st $P = 0$.

$\rightsquigarrow \underline{y = c_1 e^x + c_2 x e^x + \frac{1}{3} x^3 e^x - \frac{1}{4} e^x \sin(2x)}$ $x \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R}$

7. $y'''' - 5y'' + 4y = \sin x \cos 2x$

y_H: $\lambda^4 - 5\lambda^2 + 4 = (\lambda^2-4)(\lambda^2-1) = 0 \iff \lambda^2 = 4 \vee \lambda^2 = 1$

$\lambda = \pm 2 \qquad \lambda = \pm 1$

$\rightarrow FS = \{e^x, e^{-x}, e^{2x}, e^{-2x}\} \rightarrow y_H = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$

y_P: $f(x) = \sin x \cos 2x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\cdot \frac{1}{2}\left(e^{i2x} + e^{-i2x}\right)$

$= \frac{1}{2\cdot 2i}\left(e^{i3x} + e^{-ix} - e^{ix} - e^{-i3x}\right) = \frac{1}{2}\left(\sin(3x) - \sin x\right)$

$\lambda = 0 + 3i \qquad \lambda = 0 + i$

y_{P1}: $f_1 = \frac{1}{2}\sin(3x)$

$y_{P1} = A\cos(3x) + B\sin(3x)$

$A \to 0, \quad B \to \frac{1}{260}$

y_{P2}: $f_2 = -\frac{1}{2}\sin x$

$y_{P2} = C\cos x + D\sin x$

$C \to 0, \quad D \to -\frac{1}{20}$

$\rightarrow y(x) = c_1 e^x + c_2 e^{-x} + c_3^{2x} + c_4^{-2x} + \frac{1}{260}\sin(3x) - \frac{1}{20}\sin x \qquad x \in \mathbb{R}$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$

(2) $y'' - 2y' + y = \frac{e^x}{x} \qquad x \neq 0$

$\underline{y_H}$: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0 \Leftrightarrow \lambda_{1,2} = 1 \rightarrow y_H = c_1 e^x + c_2 x e^x$

$\underline{y_P}$: Variance konstant

$$\begin{pmatrix} e^x & x e^x \\ e^x & (1+x)e^x \end{pmatrix} \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^x}{x} \end{pmatrix} \quad \begin{matrix} \dots 1 \\ \dots 2 \end{matrix}$$

equ 2 − equ 1 : $0 \cdot c_1' = (1 + \cancel{x} - \cancel{x}) c_2' e^x = \frac{e^x}{x}$

$c_2' = \frac{1}{x} \rightarrow c_2 = \ln|x|$

$c_1' = -x c_2' = -1 \rightarrow c_1 = -x$

$\Rightarrow y_P = -x e^x + \ln|x| \, x e^x = x e^x (\ln|x| - 1)$

$\rightarrow y = y_H + y_P = c_1 e^x + x e^x (\ln|x| \underbrace{- 1 + c_2}_{\tilde{c}_2}) = c_1 e^x + x e^x (\ln|x| + \tilde{c}_2)$

9. $y'' + 4y = 2\,\mathrm{tg}\,x$ $\quad \boxed{\cos x \neq 0,\ x \neq \frac{\pi}{2} + k\pi}$ $\quad \{e^{2i}, e^{-2i}\}$

y_H: $\lambda^2 + 4 = 0 \iff \lambda_{1,2} = \pm 2i \to FS = \{\cos 2x, \sin 2x\} \to y_H = c_1 \cos 2x + c_2 \sin 2x$

y_P: Variace konstant

$$\underbrace{\begin{pmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{pmatrix}}_{W} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2\,\mathrm{tg}\,x \end{pmatrix}$$

$W \dots \det W = 2\cos^2(2x) + 2\sin^2 2x = \underline{2}$

Cramerovo pravidlo...
$$\begin{vmatrix} 0 & \sin 2x \\ 2\,\mathrm{tg}\,x & 2\cos 2x \end{vmatrix} = -2\sin 2x\,\mathrm{tg}\,x$$
$$\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & 2\,\mathrm{tg}\,x \end{vmatrix} = +2\cos 2x\,\mathrm{tg}\,x$$

$$c_1' = -\frac{2\sin 2x\,\mathrm{tg}\,x}{2} = -2\sin x\cos x\,\mathrm{tg}\,x = -2\sin^2 x = -\cancel{2}\frac{1-\cos 2x}{\cancel{2}}$$

$$\to c_1 = \int (\cos 2x - 1)\,dx = +\frac{\sin 2x}{2} - x$$

$$c_2' = +\frac{2\cos 2x\,\mathrm{tg}\,x}{2} = (\cos^2 x - \sin^2 x)\frac{\sin x}{\cos x} = (2\cos^2 x - 1)\frac{\sin x}{\cos x}$$

$$= 2\sin x\cos x - \frac{\sin x}{\cos x} = \sin 2x - \frac{\sin x}{\cos x}$$

$$\to c_2 = \int\left(\sin 2x - \frac{\sin x}{\cos x}\right)dx = -\frac{\cos 2x}{2} + \ln|\cos x|$$

$$\to y_P = \left(\cancel{+\frac{\sin 2x}{2}} - x\right)\cos 2x + \left(\ln|\cos x| - \cancel{\frac{\cos 2x}{2}}\right)\sin 2x$$

$$= -x\cos 2x + \sin 2x \ln|\cos x|$$

$$y = y_H + y_P = c_1\cos(2x) + c_2\sin(2x) - x\cos(2x) + \sin(2x)\ln|\cos x|$$

$x \in \frac{\pi}{2} + k\pi$

$c_1, c_2 \in \mathbb{R}$

(Nelze slepit v $x \in \frac{\pi}{2} + k\pi$)

(10) $x^2 y''' = 2y'$ ... pro $x \neq 0$ Eulerova rovnice

$x^3 y''' - 2xy' = 0$

reálně $y = |x|^\lambda$ → stejná char. rovnice i pro $x<0$

→ Řešení ve tvaru $y = x^\lambda$ ... $y' = \lambda x^{\lambda-1}$, $y'' = \lambda(\lambda-1)x^{\lambda-2}$, $y''' = \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$

→ ODR: $x^3 \lambda(\lambda-1)(\lambda-2)x^{\lambda-3} - 2x\lambda x^{\lambda-1} = 0$

$$\left[\lambda(\lambda-1)(\lambda-2) - 2\lambda\right] x^\lambda = 0 \qquad x \neq 0$$

$$\lambda\left((\lambda-1)(\lambda-2) - 2\right) = \lambda(\lambda^2 - 3\lambda) = \lambda^2(\lambda-3) = 0$$

$\lambda = 0$ 2-násobný, $\lambda = 3$

$x \to \ln|x|$

→ $FS = \{|x|^0, \ln|x|\,|x|^0, |x|^3\} = \{1, \ln|x|, x^3\}$

→ $y = c_1 + c_2 \ln|x| + c_3 x^3$

Substituce $\xi = \ln x$ pro $x > 0$ → $x = e^\xi$

$z(\xi) := y(x(\xi)) = y(e^\xi)$

→ $z'(\xi) = \frac{dy}{d\xi} = \frac{dy}{dx}\frac{dx}{d\xi} = y' e^\xi = y' x$

$z''(\xi) = y'' x^2 + y' x$

$z'''(\xi) = y''' x^3 + y'' x^2 + 2y'' x^2$

$z'(\xi) = y'(e^\xi)e^\xi = x y'(x)$

$z''(\xi) = y''(e^\xi)e^{2\xi} + y'(e^\xi)e^\xi$
$= x^2 y''(x) + x y'$

$z'''(\xi) = y'''(e^\xi)e^{3\xi} + y''(e^\xi)2e^{2\xi}$
$+ y''(e^\xi)e^{2\xi} + y'(e^\xi)e^\xi$
$= x^3 y'''(x) + 3x^2 y''(x) + x y'$

$z' = x y'$
$z'' = x^2 y'' + z'$
$z''' = x^3 y''' + 3z'' - 2z'$

ODR: $z''' - 3z'' + \cancel{2z'} - \cancel{2z'} = 0$

$z''' - 3z'' = 0$

$\lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3) = 0$ → FS: $\{e^{0\cdot\xi}, \xi e^{0\cdot\xi}, e^{3\xi}\}$

→ $\{1, \ln x, x^3\}$

(11) $x^2y'' + xy' + 4y = 10x$ (Euler) $\lambda = \mu + i\nu$

$|x|^\mu \cos \nu \ln|x|,\ |x|^\mu \sin \nu \ln|x|$

$(\ln|x|\,|x|^\mu \cos \nu \ln|x|, \dots)$

y_H: $y = x^\lambda \to \lambda(\lambda-1) + \lambda + 4 = 0$

$\lambda^2 - \lambda + \lambda + 4 = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i \to FS = \{\cos(2\ln|x|), \sin 2\ln|x|\}$

y_P: Mod. spec. PS:

$$f(x) = x^\mu \left(P_1(\ln x)\cos(\nu \ln x) + P_2(\ln x)\sin(\nu \ln x)\right)$$

$$\to y_P(x) = x^\mu \ln^k x \left(Q_1(\ln x)\cos(\nu \ln x) + Q_2(\ln x)\sin(\nu \ln x)\right)$$

k … násobnost $\mu + i\nu$ jako kořen char. pol.

$f(x) = 10x$ … $\mu + i\nu = 1$ … není kořen char. pol.

$\Rightarrow y_P = Ax$

$y_P' = A$

$y_P'' = 0$

ODR: $x^2 \cdot 0 + xA + 4Ax = 10x$

$5A = 10$

$A = 2 \to y_P = 2x$

$y = y_H + y_P = c_1 \cos(2\ln|x|) + c_2 \sin(2\ln|x|) + 2x$ $\quad x \in (-\infty, 0)$ nebo $(0, \infty)$

$c_1, c_2 \in \mathbb{R}$

(12) Znám jedno řešení $Ly=0$ → Najdu obecné řešení.

$$W := \det \underbrace{\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}}_{\text{Wronského matice}}$$

Wronskián

Lemma: $W'(x) = -\frac{a_{n-1}(x)}{a_n(x)} W(x)$ na (a,b) ... $Ly = a_n y^{(n)} + \dots + a_1 y' + a_0 y$

$\leadsto x_0, x \in (a,b) \Rightarrow W(x) = W(x_0)\, e^{-\int \frac{a_{n-1}(t)}{a_n(t)} dt}$

• lin ODR 2. řádu: $y'' + p(x)y' + q(x)y = 0$ (homogenní)

→ 2 lin. nezávislá řešení $u(x), v(x)$ ⇝ $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

u chodou, $u \neq 0$

v chci najít

→ $W(x) = W(x_0)\, e^{-\int \frac{p(t)}{1} dt} = uv' - u'v$

můžu vzít $W(x_0)=0$ (jen faktor násobící jedno řešení)

$u \neq 0 \Rightarrow \left(\frac{v}{u}\right)' = \frac{v'u - vu'}{u^2} = \frac{1}{u^2} e^{-\int_{x_0}^x p(t)dt}$

$\Rightarrow v(x) = u(x) \int \frac{1}{u^2(x)} e^{-\int_{x_0}^x p(t)dt} dx$

$(2x+1)y'' + 4xy' - 4y = 0$ a jedno řešení $u = e^{ax}$, $a = ?$

→ $u = e^{ax}$, $u' = ae^{ax}$, $u'' = a^2e^{ax}$ } ODR: $(2x+1)a^2 \cancel{e^{ax}} + 4xa\cancel{e^{ax}} - 4\cancel{e^{ax}} = 0$

$a_{1,2} = \frac{-4x \pm \sqrt{4^2x^2 + 4^2(2x+1)}}{2(2x+1)} = 2\,\frac{-x \pm \sqrt{(x+1)^2}}{2x+1} = 2\,\frac{-x \pm |x+1|}{2x+1}$

$= \begin{cases} \frac{2}{2x+1} \\ -2 \to e^{-2x} \end{cases}$ $\quad (2x+1)(-2)^2 + 4x(-2) - 4 = 0$ ✓

→ FS $= \{e^{-2x}, v\}$ hledám

$W' = -\frac{4x}{2x+1} W$ → $W_0(x)\, e^{-\int_{x_0}^x \frac{4t}{2t+1}dt}$ → $v = \frac{e^{-2x}}{u} \int \underbrace{\frac{1}{e^{-4x}}}_{\frac{1}{u^2}} e^{-\int_{x_0}^x \frac{4t}{2t+1}dt}$

$-\int_{x_0}^x \frac{4t+2-2}{2t+1}dt = -\int_{x_0}^x 2dt + \int_{x_0}^x \frac{2dt}{2t+1} = \left[\ln|2t+1| - 2t\right]_{x_0}^x$

→ $W = W_0\, e^{[\ln|2t+1| - 2t]_{x_0}^x} = W_0 (2x+1) e^{-2x} \frac{e^{2x_0}}{2x_0 - 1} = \tilde{W}_0 (2x+1)e^{-2x}$

→ $v = e^{-2x} \int \frac{1}{e^{-4x}} (2x+1)e^{-2x} = e^{-2x} \int (2x+1)e^{2x}$

$\overset{\text{p.p.}}{=} e^{-2x}\left(\frac{1}{2}(2x+1)e^{2x} - \int \frac{1}{\cancel{2}} \cancel{2} e^{2x}dx\right) = e^{-2x}\left(\frac{1}{2}(2x+1)e^{2x} - \frac{1}{2}e^{2x}\right)$

$= e^{-2x}(xe^{2x}) = \underline{x = v(x)}$ ⇝ FS $= \{e^{-2x}, x\}$

(13) $xy'' + 2y' - xy = 0$, jedno řešení $y = \frac{e^x}{x}$

$\rightsquigarrow$ $FS = \left\{ \frac{e^x}{x} , v \right\}$

$v = u \int \frac{1}{u^2} e^{-\int_{x_0}^{x} \frac{2}{t} dt} dx$ (see problem 12)

$e^{-\int_{x_0}^{x} \frac{2}{t} dt} = e^{\left[-2\ln|t|\right]_{x_0}^{x}} = e^{\left[\ln \frac{1}{t^2}\right]_{x_0}^{x}} = e^{\ln\frac{1}{x^2} - \ln\frac{1}{x_0^2}} = \frac{x_0^2}{x^2}$. můžu vzít $:= 1$

$\frac{v}{u} = \int \frac{x^2}{e^{2x}} \frac{1}{x^2} dx = \int e^{-2x} dx = -\frac{1}{2} e^{-2x}$

$v = \frac{e^x}{x}\left(-\frac{1}{2} e^{-2x}\right) = -\frac{1}{2} \frac{e^{-x}}{x} \rightsquigarrow \tilde{v} = \frac{e^{-x}}{x}$

$\rightsquigarrow FS = \left\{ \frac{e^x}{x} , \frac{e^{-x}}{x} \right\} \rightsquigarrow y = c_1 \frac{e^x}{x} + c_2 \frac{e^{-x}}{x}$ $\quad x \neq 0$

$c_1, c_2 \in \mathbb{R}$

(14) $(x+1)xy'' + (x+2)y' - y = x + \frac{1}{x}$, řešení $y = x+2$ $\quad \underline{x \neq 0}$

$\rightarrow$ FS homogenní rovnice $\{x+2, v\} = \{u, v\}$

$\rightarrow v = u \int \left(\frac{1}{u^2} e^{-\int_{x_0}^{x} \frac{t+2}{t(t+1)} dt} \right) dx$

- $-\int_{x_0}^{x} \frac{t+2}{t(t+1)} dt = -\int_{x_0}^{x} \left(\frac{2}{t} - \frac{1}{t+1} \right) dt = \left[\ln \frac{|t+1|}{t^2} \right]_{x_0}^{x}$

- $e^{-\int_{x_0}^{x} \frac{t+2}{t(t+1)} dt} = \frac{x+1}{x^2} \underbrace{\frac{x_0^2}{x_0+1}}_{W_0 = 1 \text{ (můžu)}}$ $\qquad \cancel{x^2} + 4x + 4 - \cancel{x^2} = 4(x+1)$

- $\frac{v}{u} = \int \frac{1}{(x+2)^2} \frac{x+1}{x^2} dx = \int \left(\frac{1}{4x^2} - \frac{1}{4(x+2)^2} \right) dx = \frac{1}{4} \left(-\frac{1}{x} + \frac{1}{x+2} \right)$

$= \frac{1}{4} \left(\frac{1}{x+2} - \frac{1}{x} \right) = \frac{1}{4} \frac{\cancel{x} - \cancel{x} - 2}{x(x+2)} = \frac{1}{2} \frac{1}{x(x+2)} \rightarrow v = \cancel{(x+2)} \frac{1}{2} \frac{1}{x\cancel{(x+2)}}$

$\tilde{v} = \frac{1}{x}$

$\rightarrow FS = \{u, v\} = \{x+2, \frac{1}{x}\} \qquad y_H = c_1(x+2) + c_2 \frac{1}{x}$

Partikulární řešení $Ly = f(x)$ $\rightarrow$ Variace konstant

$$\begin{pmatrix} x+2 & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \left(x + \frac{1}{x}\right) \frac{1}{x(x+1)} \end{pmatrix} = \begin{matrix} 0 \\ \frac{x^2+1}{x^2(x+1)} \end{matrix}$$

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x)/a_2 \end{pmatrix}$$

$c_1'(x+2) + c_2' \frac{1}{x} = 0 \quad \rightarrow c_2' = -c_1' x(x+2) \quad \frac{x+2}{x}$

$c_1' \quad - c_2' \frac{1}{x^2} = \frac{x^2+1}{x^2(x+1)} \quad \ldots c_1' \left(1 + \cancel{\ldots} \right) - c_1' \frac{2(x+1)}{x} = \frac{x^2+1}{x^2(x+1)}$

$\rightarrow c_1' = \frac{1}{2} \frac{x^2+1}{x(x+1)^2} = \frac{1}{2x} - \frac{1}{(x+1)^2} \quad \rightarrow c_1 = \frac{1}{2} \ln|x| + \frac{1}{x+1}$

$c_2' = -c_1' x(x+2) = -\frac{1}{2} \frac{(x^2+1)(x+2)}{(x+1)^2} = -\frac{x}{2} - \frac{1}{(x+1)^2} \rightarrow c_2 = -\frac{x^2}{4} + \frac{1}{x+1}$

$\rightarrow y_P = c_1(x+2) + c_2 \frac{1}{x} = \left(\frac{1}{2} \ln|x| + \frac{1}{x+1} \right)(x+2) + \left(-\frac{x^2}{4} + \frac{1}{x+1} \right) \frac{1}{x}$ $\qquad \frac{x+1}{x}$

$= \frac{x+2}{x+1} + \frac{x+2}{2} \ln|x| - \frac{x}{4} + \frac{1}{x(x+1)} = -\frac{x}{4} + \frac{x+2}{2} \ln|x| + \frac{x^2+2x+x}{x(x+1)}$

$\rightarrow y = y_H + y_P = c_1(x+2) + \frac{c_2}{x} - \frac{x}{4} + \frac{x+2}{2} \ln|x| + \frac{x+1}{x}$ $\qquad x \neq 0$

$c_1, c_2 \in \mathbb{R}$

(15) $(2x+1)y'' + (2x-1)y' - 2y = x^2 + x$

$FS = \{u, v\} = \{\underset{P_n}{\text{polynom}}, v\}$ " $Ly = \tilde{P}_1 \underset{P_{n-1}}{P_{n-2}} + \tilde{\tilde{P}}_1 \underset{P_n}{P_{n-1}} - \underset{P_n}{P_n}$ "

$y = a_n x^n + \dots \quad x^n: \underbrace{2\cancel{x} a_n n \cancel{x^{n-1}}}_{(2x-1)y'} - \underbrace{2a_n x^n}_{2y} = 0 \dots n-1=0 \rightarrow n=1$

$\rightarrow u = ax+b$, $u' = a$, $u'' = 0$ $\rightarrow$ ODR $(2x+1)\cdot\phi + (2x-1)a - 2(ax+b) = 0$

$x^1: 2a - 2a = 0$ ✓

$x^0: -a - 2b = 0 \quad a = -2b \rightarrow u = 2x-1$

$\rightarrow FS = \{u, v\} = \{2x-1, v\}$

$\rightarrow v = u \int \frac{1}{u^2} e^{-\int_{x_0}^{x} \frac{2t-1}{2t+1} dt} du \quad \dots x \neq \frac{1}{2}$

$\frac{w'}{w} = -\frac{2x-1}{2x+1} \quad \dots -\int_{x_0}^{x} \frac{2t-1}{2t+1} dt = -\int_{x_0}^{x} \left(1 - \frac{2}{2t+1}\right) dt = \left[\ln|2t+1| - t\right]_{x_0}^{x}$

$w = w_0 e^{-\int_{x_0}^{x} \frac{2t-1}{2t+1} dt} = w_0 (2x+1) e^{-x} \frac{e^{x_0}}{2x_0+1} = \tilde{w}_0 (2x+1) e^{-x}$

$\frac{v}{u} = \int \frac{1}{(2x-1)^2} (2x+1) e^{-x} dx =$

$= \frac{e^{-x}}{1-2x} \rightarrow v = \underbrace{(1-2x)}_{u} \frac{e^{-x}}{1-2x} = e^{-x}$

$\rightarrow FS = \{u, v\} = \{2x-1, e^{-x}\} \rightarrow \underline{y_H = c_1(2x-1) + c_2 e^{-x}}$

y_P → Variace konstant
y_P → Hledání $y_P = ax^2 + bx + c$

$y_P' = 2ax + b$

$y_P'' = 2a$

$\rightarrow (2x+1)2a + (2x-1)(2ax+b) - 2(ax^2+bx+c) = x^2 + x$

$x^0: 2a - b - 2c = 0 \rightarrow 1 - b - 2c = 0 \rightarrow 1 = b + 2c$

$x^1: 4a + 2b - 2a - 2b = 1 \rightarrow 2a = 1 \rightarrow a = \frac{1}{2}$ ✓

$x^2: 4a - 2a = 1 \rightarrow 2a = 1 \rightarrow a = \frac{1}{2}$

$\left(\begin{cases} b = -1 \\ c = 1 \end{cases}\right.$ ← jednoduchá

$\rightarrow y_P = \frac{x^2}{2} - x + 1$

$\rightarrow y = y_H + y_P = c_1(2x-1) + c_2 e^{-x} + \frac{x^2}{2} - x + 1 \qquad x \neq \frac{1}{2}$ (?)

$c_1, c_2 \in \mathbb{R}$

(16) $2yy' = y^2 + y'^2$

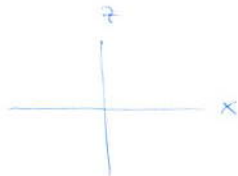
$y'^2 - 2yy' + y^2 = 0$

$(y' - y)^2 = 0$

$y' - y = 0 \rightarrow \lambda - 1 = 0 \rightarrow y = ce^x$

(17) $x^2 y'' = y'^2 \rightarrow y'' = \frac{y'^2}{x^2}$ pro $x \neq 0$ typ $y^{(n)} = f(x, y^{(n-1)})$

- $y = \text{const.}$ triviální řešení
- $z = y' \rightarrow z' = y'' \rightarrow z' = \frac{z^2}{x^2}$ ...



- $z = 0$ triviální ř.
- $\frac{z'}{z^2} = \frac{1}{x^2}$

$\hookrightarrow -\frac{1}{z} = -\frac{1}{x} - c_1 \rightarrow \frac{1}{z} = \underbrace{\frac{1}{x} + c_1}_{\frac{1}{x} \neq -c_1} \rightarrow z = \frac{1}{\frac{1}{x} + c_1} = \frac{x}{1 + xc_1}$

- $y' = z = \frac{x}{1 + xc_1}$

$y = \int z\,dx = \int \frac{x}{1 + xc_1}\,dx = \frac{1}{c_1}\int \frac{xc_1 + 1 - 1}{xc_1 + 1}\,dx = \frac{1}{c_1}\int \left(1 - \frac{1}{1 + xc_1}\right)dx$

$= \frac{1}{c_1}\left(x - \frac{1}{c_1}\ln|1 + xc_1|\right) = \frac{x}{c_1} - \frac{\ln|1 + c_1 x|}{c_1^2} + c_2$

(12) $y^3 y'' = 1 \quad \dots \quad y'' = \dfrac{1}{y^3} \qquad y \neq 0$

typ $y^{(n)} = f\left(y^{(n-2)}\right)$

nejčastěji pouze $\underline{n = 2}$

$\Big/ \cdot 2y'$

$$2y'y'' = \frac{2y'}{y^3}$$

$$\left(y'^2\right)' = \left(-\frac{1}{y^2}\right)' \;\rightarrow\; y'^2 = -\frac{1}{y^2} + c_1$$

$$y' = \pm\sqrt{c_1 - \frac{1}{y^2}} \quad \dots \quad c_1 - \frac{1}{y^2} \geq 0 \quad \Leftrightarrow \quad \frac{1}{y^2} > c \quad \Leftrightarrow \quad y^2 < \frac{1}{c} \qquad c > 0$$

$$y \in \left[-\frac{1}{\sqrt{c}}, +\frac{1}{\sqrt{c}}\right]$$

$\vdots$

(19) $y'' = e^y \quad \Big| \cdot 2y'$

$$2y'y'' = 2y'e^y$$

$$(y'^2)' = (2e^y)' \rightarrow y'^2 = 2\underbrace{(e^y + c)}_{}$$

$e^y + c \geq 0$

$e^y \geq -c$ ... pro $c<0$

$y \geq \ln(-c)$

$$y' = \pm\sqrt{2}\sqrt{e^y + c}$$

$$\pm\frac{y'}{\sqrt{e^y+c}} = \sqrt{2}$$

$$\pm\int\frac{dy}{\sqrt{e^y+c}} = \int\sqrt{2}\,dx = \sqrt{2}(x + c_2)$$

$$\left|\begin{array}{l} e^y = u \\ e^y dy = du \\ dy = \frac{du}{u}\end{array}\right| = \pm\int\frac{du}{u\sqrt{u+c}} \left|\begin{array}{l}\sqrt{u+c} = v \\ \frac{1}{2}\frac{du}{\sqrt{u+c}} = dv \\ \frac{1}{2}du = v\,dv \\ u + c = v^2 \\ u = v^2 - c\end{array}\right| = \pm\int\frac{2\cancel{v}\,dv}{(v^2-c)\cancel{v}}$$

$$= \pm 2\int\frac{1}{v^2-c}dv = \mp\frac{2}{c}\underbrace{\int\frac{1}{1-\frac{v^2}{c}}dv}_{\operatorname{argtanh}\left(\frac{v}{\sqrt{c}}\right)\sqrt{c}}$$

$$= \mp\frac{2}{\sqrt{c}}\operatorname{argtanh}\left(\frac{v}{\sqrt{c}}\right)$$

$$\sqrt{2}(x + c_2) = \mp\frac{2}{\sqrt{c}}\operatorname{argtanh}\frac{v}{\sqrt{c}}$$

$$= \mp\frac{2}{\sqrt{c}}\operatorname{argtanh}\sqrt{\frac{u+c}{c}}$$

$$= \mp\frac{2}{\sqrt{c}}\operatorname{argtanh}\sqrt{\frac{e^y}{c}+1}$$

(20) $y'' + y'^2 = 2e^{-y}$ ... typ $y^{(n)} = f(y, y', \dots, y^{n-1})$
$n = 2$

$y' = p(y(x)) = p(z)$

$y'' = p'(y(x))\, y' = p'(y(x))\, p(y(x)) = p'(z)p(z)$

$p'(z)p(z) + p^2(z) = 2e^{-z}$ $\quad p(z) \neq 0$

$p'(z) + p(z) = \frac{2e^{-z}}{p(z)}$ ... Bernoulli, $\alpha = -1$

~~$\lambda + 1 = 0 \to \lambda = -1$~~

$\eta(z) = p(z)^{1-\alpha} = p(z)^2 \to \eta' = 2pp'$

$2pp' + 2p^2 = 4e^{-z} \to \underline{\eta' + 2\eta = 4e^{-z}}$

η_H: $\lambda + 2 = 0 \to \eta_H = c_1 e^{-2z}$

η_P: $\eta_P = c_1(z) e^{-2z}$

$\eta_P' = c_1' e^{-2z} + c_1(-2)e^{-2z}$

ODE: $c_1' e^{-2z} - \cancel{2c_1 e^{-2z}} + \cancel{2c_1 e^{-2z}} = 4e^{-z}$

$c_1' = 4e^{z} \to c_1 = \int 4e^{z}\,dz = 4e^{z}$

$\to \eta_P = 4e^{z}e^{-2z} = \underline{4e^{-z}}$

$\eta(z) = p^2(z)$

$\eta(z) = \eta_H + \eta_P = c_1 e^{-2z} + 4e^{-z} > 0$

$\hookrightarrow c_1 > -4e^{z}$

$-\frac{c_1}{4} < e^{z}$

$\ln\left(-\frac{c_1}{4}\right) < z$ pro $c_1 < 0$

$\eta = p^2 = e^{-2z}(c_1 + 4e^{z}) \to p(z) = \pm e^{-z}\sqrt{c_1 + 4e^{z}}$

$p(z) = p(y(x)) = y'(x) \to y' = \pm e^{-y}\sqrt{c_1 + 4e^{y}}$

$\frac{y'}{\pm e^{-y}\sqrt{c_1 + 4e^{y}}} = 1$

- stac. řeš. $y = \ln\left(-\frac{c_1}{4}\right)$ $(c<0)$
- $y = \ln\left[(x+c_2)^2 - \frac{c_1}{4}\right]$
  $z > \ln\left(-\frac{c_1}{4}\right)$ $(c<0)$
  $z \in \mathbb{R}$ $(c>0)$

Pro $c \neq -4e^{y}$ (může nastat pro $c<0$): $\int \frac{e^{y}}{\sqrt{c_1 + 4e^{y}}}\,dy \;\Big|\; u = e^{y},\ du = e^{y}dy \;\Big|\; = \int \frac{du}{\sqrt{c_1 + 4u}}$

$= \frac{1}{2}\sqrt{c_1 + 4u} = \int 1\,dx = x + c_2 \to \sqrt{c_1 + 4e^{y}} = 2(x + c_2)$

$c_1 + 4e^{y} = 4(x+c_2)^2 \to e^{y} = (x+c_2)^2 - \frac{c_1}{4}$

$y = \ln\left[(x+c_2)^2 - \frac{c_1}{4}\right]$