Sada příkladů 2/3

Obyčejné diferenciální rovnice

Lineární rovnice 1. řádu

1. $y' \cos x = y \sin x + \cos^2 x$
2. $y' - 2\dfrac{y}{x} = x^3$
3. $y' + 2xy = 2x\mathrm{e}^{-x^2}$
4. $y' + y \sin x = \sin x \cos x$
5. $xy' + y = \ln x + 1$
6. $(2\mathrm{e}^y - x)y' = 1$. (Hledejte řešení ve tvaru $x = x(y)$.)
7. Najděte právě to řešení rovnice $y' \sin 2x = 2(y + \cos x)$, které je omezené pro $x \to \frac{\pi}{2}$.

Bernoulliova rovnice

8. $xy' - 2x^2\sqrt{y} = 4y$
9. $y' - 2xy = 2x^3y^2$
10. $y' - \dfrac{1}{x}y = \dfrac{1}{2y}$
11. $xy' + y = y^2 \ln x$, $y(1) = 1$
12. $y' - xy = -y^3\mathrm{e}^{-x^2}$
13. $y' - 9x^2y = (x^5 + x^2)y^{\frac{2}{3}}$, $y(0) = 0$.

$\boxed{\text{Lineární ODR. 1. řádu}}$

$$y'(x) + p(x)y(x) = f(x)$$

$$+\ y(x_0) = y_0$$

Metoda integračního faktoru: $e^{\int p(x)dx}$

$$\rightarrow \left(y e^{\int p(x)dx}\right)' = y' e^{\int p(x)dx} + p(x) y e^{\int p(x)dx} = f(x) e^{\int p(x)dx}$$

$$\rightarrow y e^{\int p(x)dx} = \int f(x) e^{\int p(x)dx} dx + c$$

$$\rightarrow y = e^{-\int p(x)dx}\left[\int f(x) e^{\int p(x)dx} dx + c\right]$$

Metoda variace konstant

1. y_H: $y_H' + p(x) y_H = 0 \rightarrow y_H = K e^{-\int p(x)dx}$
2. y_P: jedno partikulární řešení: metodou variace konstant $y_P = c(x) e^{-\int p(x)dx}$

   Dosadím do ODR s pravou stranou $f(x)$ a najdu $c(x)$

3. $y = y_P + y_H$

$\boxed{\text{Bernoulliova rovnice}}$

$$y'(x) + p(x)y(x) = f(x)y^{\alpha}(x) \quad + \quad y(x_0) = y_0$$

- $\alpha = 0$ a $\alpha = 1$ ... lineární ODR 1. řádu
- $\alpha < 0$ ... $y \neq 0$ musí platit → netriviální řešení pro $y \in (-\infty, 0)$ a $(0, +\infty)$
- $\alpha > 0$ a $\alpha \neq 1$ ... triviální řešení $y = 0$

  netriviální řešení pro $y \in (-\infty, 0)$ a $(0, +\infty)$

$$\rightarrow y' y^{-\alpha} + p(x) y^{1-\alpha} = f(x)$$

Definujeme $z(x) = y^{1-\alpha}(x) \rightarrow z' = (1-\alpha) y' y^{-\alpha} \Rightarrow \underbrace{\frac{z'}{1-\alpha} + p(x) z = f(x)}_{\text{lineární ODR 1. řádu}}$

① $y' \cos x = y \sin x + \cos^2 x$

$x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \rightarrow 0 = \pm y$

$x \neq \frac{\pi}{2} + k\pi \rightarrow y' - \operatorname{tg} x \, y = \cos x$

$y'(x) + p(x)\, y(x) = f(x)$

Integrační faktor

$P(x) = \int p(x)\,dx = -\int \operatorname{tg} x \,dx = -\int \frac{\sin x}{\cos x}\,dx \;\Big|\; \begin{matrix} \cos x = u \\ -\sin x\,dx = du \end{matrix} \Big| = +\int \frac{du}{u} = \ln|u| + c$

$= \ln|\cos x| + c$

$\rightarrow e^{P(x)} = \cos x$

$\cos x\, y' - \sin x\, y = (\cos x\, y)' = \cos^2 x$

$\cos x\, y = \int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos(2x))\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + c$

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1 + \cos(2x)}{2}$

$\sin(2x) = 2 \sin x \cos x$

$= \frac{x}{2} + \frac{1}{2}\sin x \cos x + c$

$\rightarrow \underline{y = \frac{1}{2}\sin x + \frac{x}{2\cos x} + \frac{c}{2\cos x}}, \quad x \neq \frac{\pi}{2} + k\pi$

Lineární → variace konstant, $y = y_H + y_P$

y_H: $y' - \operatorname{tg} x\, y = 0 \quad x \neq \frac{\pi}{2} + k\pi$

$\frac{y'}{y} = \operatorname{tg} x$

$\ln|y| = -\ln|\cos x| + \ln|K|$

$y_H = \frac{c}{\cos x}$

y_P: $y = \frac{c(x)}{\cos x} \rightarrow \left(\frac{c(x)}{\cos x}\right)' - \frac{\sin x}{\cos x}\frac{c(x)}{\cos x} = \cos x$

$\rightarrow \frac{c'}{\cos x} - \cancel{\frac{\sin x\, c}{\cos^2 x}} = \cos x \rightarrow c' = \cos^2 x$

$- \cancel{\frac{c}{\cos^2 x}(-\sin x)}$

$c(x) = \int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{2}\sin x \cos x$

$\rightarrow y_P = \frac{x}{2\cos x} + \frac{1}{2}\sin x$

$\rightarrow y = y_H + y_P$

$= \underline{\frac{1}{2}\sin x + \frac{x}{2\cos x} + \frac{c}{\cos x}}$

② $y' - 2\frac{y}{x} = x^3 \qquad x \neq 0$

Metoda integračního faktoru

$p(x) = -\frac{2}{x} \rightarrow P(x) = \int p(x)\,dx = -\int \frac{2}{x}\,dx = -2\ln|x| = \ln\frac{1}{x^2}$

$e^{P(x)} = \frac{1}{x^2}$

$$\frac{y'}{x^2} - \frac{2y}{x^3} = \left(\frac{y}{x^2}\right)' = x$$

$$\frac{y}{x^2} = \int x\,dx = \frac{x^2}{2} + c$$

$$y(x) = \frac{x^4}{2} + cx^2 \qquad x \neq 0$$

$c \in \mathbb{R}$

$\lim_{x\to 0} y(x) = 0$

$\lim_{x\to 0} y'(x) = 0$

$\rightarrow xy' - 2y = x^4$

$\hookrightarrow y(x) = \frac{x^2}{2} + cx^2 \quad x \in \mathbb{R}$

$y = y_H + y_P$, variace konstant

y_H: $y' = \frac{2y}{x}$

$\frac{y'}{y} = \frac{2}{x} \rightarrow \ln|y| = 2\ln|x| + \ln|c| = \ln|c|x^2$

$\rightarrow \underline{y_H = cx^2}$

y_P: $y = c(x)x^2 \rightarrow$

$c'x^2 + \cancel{c2x} - \cancel{\frac{2cx^2}{x}} = x^3$

$c' = x \rightarrow c = \frac{x^2}{2} + c_0 \rightarrow y_P = \frac{x^4}{2}$

but we don't care about it here

$$\rightarrow y = y_H + y_P = \frac{x^4}{2} + cx^2 \qquad x \neq 0$$

3 $y' + 2xy = 2xe^{-x^2}$

Integráló faktor

$p(x) = 2x \quad P(x) = \int 2x\,dx = x^2 \to e^{P(x)} = e^{x^2}$

$$e^{x^2} y' + 2xe^{x^2} y = \left(e^{x^2} y\right)' = 2x\cancel{e^{-x^2} e^{x^2}} = 2x$$

$$\to e^{x^2} y = \int 2x\,dx = x^2 + C$$

$$\to y = e^{-x^2}\left(x^2 + C\right) \quad x \in \mathbb{R}$$

4) $y' + y \underbrace{\sin x}_{p(x)} = \sin x \cos x$

Integr. faktor

$$P(x) = \int p(x)\,dx = \int \sin x\,dx = -\cos x \;(+C) \rightarrow e^{P(x)} = e^{-\cos x}$$

$$e^{-\cos x} y' + \sin x\, e^{-\cos x} y = \left(e^{-\cos x} y\right)' = \sin x \cos x\, e^{-\cos x}$$

$$\cancel{\rightarrow e^{-\cos x} y = \int \sin x \cos x\,dx \left| \begin{array}{l} \sin x = u \\ \cos x\,dx = du \end{array} \right| = \int u\,du = \frac{u^2}{2} = \frac{\sin^2 x}{2} + C}$$

$$\cancel{\rightarrow y = e^{\cos x}}$$

$$\rightarrow e^{-\cos x} y = \int \cos e^{-\cos x} \sin x\,dx \left| \begin{array}{l} \cos x = u \\ \sin x\,dx = du \end{array} \right| = -\int u e^{-u}\,du$$

$$= +u e^{-u} - \int e^{-u}\,du = +(1+u)e^{-u} + c = +(1+\cos x)e^{-\cos x}$$

$$\rightarrow y = 1 + \cos x + c e^{\cos x}$$

(5) $xy' + y = \ln x + 1 \qquad x \in (0, \infty)$

$$y' + \frac{y}{x} = \frac{\ln x + 1}{x}$$

Integrační faktor

$$p(x) = \frac{1}{x} \to P(x) = \int \frac{1}{x}\,dx = \ln x \quad (+c_0) \quad \to e^{P(x)} = e^{\ln x} = x$$

$$\to \quad xy' + y = (xy)' = \ln x + 1$$

$$\to xy = \int (\ln x + 1)\,dx = x\ln x \cancel{- x} \cancel{+ x} + c = x\ln x + c$$

$$y = \ln x + \frac{c}{x} \qquad x > 0$$

(6) $(2e^y - x)y' = 1$ Hledejme řešení ve tvaru $x = x(y)$

$\frac{d}{dy}$

$x' = 2e^y - x \quad \to \quad x' + x = 2e^y$

$p(y) = 1 \to P(y) = \int 1\,dy = y \ (+c_0) \quad \to e^{P(y)} = e^y$

$$e^y x' + e^y x = (e^y x)' = 2e^{2y}$$

$$\to e^y x = \int 2e^{2y}\,dy = e^{2y} + c$$

$$\to \quad x = e^y + \frac{c}{e^y} \quad y \in \mathbb{R}$$

(7) Najděte řešení rovnice $y' \sin(2x) = 2(y+\cos x)$, které je omezené pro $x \to \frac{\pi}{2}$

$$y'\sin(2x) - 2y = 2\cos x$$

$n \in \mathbb{Z}$

$2x = k\pi \iff x = k\frac{\pi}{2} \Rightarrow y = -\cos x$
- 1 $k = 0,4,8$ $4n$
- -1 $k = 2,6,10$ $4n+2$
- 0 $k = 1,3,5$ $2n+1$

$x \neq k\frac{\pi}{2}$

$$y' - \frac{2y}{\sin(2x)} = 2\frac{\cos x}{\sin(2x)}$$

$$p(x) = -\frac{2}{\sin(2x)} \to P(x) = -\int \frac{2}{\sin(2x)}\,dx = -\int \frac{\cancel{2}}{\cancel{2}\sin x\cos x}\,dx$$

$$= -\int \frac{\cos dx}{\sin x \cos^2 x} \quad \left| \begin{matrix} u = \operatorname{tg} x \\ du = \frac{dx}{\cos^2 x} \end{matrix} \right| = -\int \frac{du}{u}$$

$$= -\ln|u| + c = -\ln|\operatorname{tg} x| (+c) \to e^{P(x)} = e^{-\ln|\operatorname{tg} x|} = \frac{1}{\operatorname{tg} x}$$

$$\frac{\cos x}{\sin x} y' - \frac{\cos x}{\sin x}\frac{y}{\cancel{\cos x}\sin x} = (\operatorname{cotg}(x)\, y)' = \frac{\cos x}{\sin x \sin x \cancel{\cos x}}$$

$$= \frac{\cos x}{\sin^2 x}$$

$$\operatorname{cotg}(x)\, y = \int \frac{\cos x\,dx}{\sin^2 x} \quad \left| \begin{matrix} \sin x = v \\ \cos x\,dx = dv \end{matrix} \right| = \int \frac{dv}{v^2} = -\frac{1}{v} + c$$

$$= -\frac{1}{\sin x} + c$$

$$y = -\frac{1}{\cos x} + c\,\operatorname{tg}(x) \quad \dots x \neq k\frac{\pi}{2}$$

$\operatorname{tg}(x) \xrightarrow{x \to \frac{\pi}{2}} +\infty$

$$\Rightarrow c = 0 : \quad y = -\frac{1}{\cos x}$$

(8) $xy' - 2x^2\sqrt{y} = 4y$

Bernoulli: $y'(x) + p(x)y(x) = f(x)y^\alpha$

$\alpha \neq 0, 1$

$x = 0 \to 4y = 0$

$x \neq 0 \to y' - \frac{4y}{x} = 2x\sqrt{y}$ ... Bernoulli $\alpha = \frac{1}{2}$

- Triviální řešení: $y = 0$
- Netriviální: $(y \neq 0)$ $\frac{y'}{\sqrt{y}} - \frac{4\sqrt{y}}{x} = 2x$

$z(x) = y^{1-\alpha}(x) = y^{1-\frac{1}{2}}(x) = \sqrt{y} \to z' = \frac{1}{2}\frac{y'}{\sqrt{y}}$

$\to 2z' - \frac{4z}{x} = 2x \to z' \underbrace{- \frac{2}{x}}_{\tilde{p}(x)} z = x$

Integrační faktor

$\tilde{p}(x) = -\frac{2}{x} \to \tilde{P}(x) = -\int \frac{2}{x}dx = -2\ln|x| = \ln\frac{1}{x^2} \to e^{\tilde{P}(x)} = \frac{1}{x^2}$

$\frac{z'}{x^2} - \frac{2z}{x^3} = \left(\frac{z}{x^2}\right)' = \frac{1}{x}$

$\frac{z}{x^2} = \int \frac{1}{x}dx = \ln|x| + c = \ln|x| + c$

$= z = x^2\ln|x| + cx^2$

$\tilde{c} > 0$

$\sqrt{y} = x^2\underbrace{(\ln|x| + c)} = x^2\ln(\tilde{c}|x|)$

$\ln|x| + c > 0$

$c > -\ln|x|$

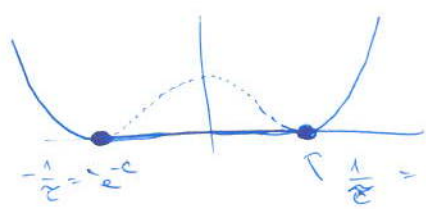
$-c < \ln|x|$

$0 < e^{-c} < |x|$

$x \in (-\infty, -e^{-c}) \cup (e^{-c}, \infty)$

$= x^4\ln^2\tilde{c}|x|$

$y = \begin{cases} x^4(\ln|x| + c)^2 & \text{na } (-\infty, -e^{-c}) \cup (e^{-c}, \infty) \\ 0 & \text{na } [-e^{-c}, e^{-c}] \text{ ... lze slepit} \end{cases}$



(9) $y' - 2xy = 2x^3y^2$ ... Bernoulli, $\alpha = 2$

- Triviální: $y = 0$
- Netriviální ($y \neq 0$): $\frac{y'}{y^2} - \frac{2x}{y} = 2x^3$

$\boxed{z = y^{1-\alpha} = y^{1-2} = \frac{1}{y}} \Rightarrow z' = -\frac{y'}{y^2}$

$\rightarrow -z' - 2xz = 2x^3$

$z' + 2xz = -2x^3$

$\tilde{p}(x) = 2x \rightarrow P(x) = \int 2x\,dx = x^2 \ (+c) \rightarrow e^{P(x)} = e^{x^2}$

$e^{x^2}z' + 2xe^{x^2}z = (e^{x^2}z)' = -2x^3e^{x^2}$

$\rightarrow (e^{x^2}z) = -\int 2x^3e^{x^2} = -\int 2xe^{x^2}x^2\,dx$

$= -e^{x^2}\cdot x^2 + \int 2xe^{x^2}\,dx = -e^{x^2}x^2 + e^{x^2} = (1-x^2)e^{x^2} + c$

$\rightarrow z = 1 - x^2 + ce^{-x^2}$

$\rightarrow y = \frac{1}{z} = (1-x^2+ce^{-x^2})^{-1}$ ... $1-x^2+ce^{-x^2} \neq 0$

$\rightarrow$ $y = 0$ na $x \in \mathbb{R}$

$y = (1-x^2+ce^{-x^2})^{-1}$ na intervalech, kde $1-x^2+ce^{-x^2} \neq 0$

(10) $y' - \frac{y}{x} = \frac{1}{2y}$ Bernoulli $\alpha = -1$ $x \neq 0,\ y \neq 0$

→ $z = y^{1-\alpha} = y^2$ ... musí být $z > 0$

$z' = 2yy'$

→ $\underbrace{2yy'}_{z'} - \frac{2y^2}{x} = 1$ → $z' - \frac{2z}{x} = 1$

Integrační faktor: $p(x) = -\frac{2}{x}$ → $P(x) = \int -\frac{2}{x}\,dx = -2\ln|x| = \ln\frac{1}{x^2}$ → $e^{P(x)} = \frac{1}{x^2}$

→ $\frac{z'}{x^2} - \frac{2z}{x^3} = \left(\frac{z}{x^2}\right)' = \frac{1}{x^2}$

$$\frac{z}{x^2} = \int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$$

$$z = \underbrace{-x + cx^2} = x(cx-1)$$

$cx^2 - x > 0$

$cx^2 > x$

$c > \frac{x}{x^2}$ $x > 0:$ $x > \frac{1}{c}$

$c > \frac{1}{x}$ $x < 0:$ $x < \frac{1}{c}$

$y^2 = x(cx-1)$

$y = \pm\sqrt{x(cx-1)}$ na $\left(-\infty, \min\left(0, \frac{1}{c}\right)\right)$ a $\left(\max\left(0, \frac{1}{c}\right), +\infty\right)$

(11) $xy' + y = y^2 \ln x \quad , \quad y(1) = 1 \qquad$ Bernoulli $\alpha = 2$

$$z = y^{1-\alpha} = y^{-1} = \frac{1}{y} \to z' = -\frac{y'}{y^2} \quad \dots \quad y' = -\frac{z'}{z^2}$$

$$\to x\frac{y'}{y^2} + \frac{1}{y} = \ln x \to -xz' + z = \ln x$$

$$z' - \frac{z}{x} = -\frac{\ln x}{x}$$

$$\to e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = e^{\ln\frac{1}{|x|}} = \frac{1}{x}$$

$$\to \frac{z'}{x} - \frac{z}{x^2} = \left(\frac{z}{x}\right)' = -\frac{\ln x}{x^2}$$

$$\to \frac{z}{x} = -\int \frac{\ln x}{x^2}dx \overset{\text{per partes}}{=} \frac{\ln x}{x} - \int \frac{1}{x^2}dx = \frac{\ln x}{x} + \frac{1}{x} + c$$

$$\to z = 1 + \ln x + cx$$

$$y = \underbrace{\frac{1}{1+\ln x + cx}}_{\text{where denominator} \neq 0} \quad \dots \text{ poč. podm.: } y(1) = 1 \Leftrightarrow 1 = \frac{1}{1+c} \Leftrightarrow c = 0$$

$$\to y = \frac{1}{1+\ln x} \quad \dots \quad \left.\begin{array}{l} 1 + \ln x \neq 0 \Leftrightarrow \ln x \neq -1 \\ x = e^{-1} < 1 \end{array}\right\} \to \text{na } \left(\frac{1}{e}, +\infty\right)$$

(12) $y' - xy = -y^3 e^{-x^2}$ Bernoulli, $\alpha = 3$ $y = 0$ triv. řešení

Substituce $z = y^{1-3} = y^{-2} = \frac{1}{y^2}$ ... $z' = -\frac{2y'}{y^3}$

$\to -\frac{2y'}{y^3} + 2x\frac{1}{y^2} = +2e^{-x^2} \to z' + 2xz = 2e^{-x^2}$

$\to$ Uhodnu integrační faktor: $(e^{x^2} z)' = 2$

$\to e^{x^2} z = \int 2\,dx = 2x + c$

$\frac{1}{y^2} = z = \frac{2x+c}{e^{x^2}} \leftarrow 2x + c > 0 \Leftrightarrow x > -\frac{c}{2}$

$y^2 = \frac{e^{x^2}}{2x+c}$

$y = \pm \frac{e^{\frac{x^2}{2}}}{\sqrt{2x+c}}$ na $\left(-\frac{c}{2}, +\infty\right)$

$y = 0$ na $\mathbb{R}$

(13) $y' - 9x^2y = (x^5+x^2)\,y^{2/3}\,,\quad y(0)=0$

Bernoulli, $\alpha = \frac{2}{3}$, substituce $z = y^{1-\alpha} = y^{1/3} \Rightarrow z' = \frac{1}{3}\frac{y'}{y^{2/3}}$

$$\frac{1}{3}\frac{y'}{y^{2/3}} - \frac{3x^2}{y^{1/3}} = \frac{x^5+x^2}{3}$$

$$z' - 3x^2 z = \frac{x^5+x^2}{3}$$

$$e^{P(x)} = e^{\int -3x^2\,dx} = e^{-x^3} \rightarrow \left(e^{-x^3} z\right)' = \frac{1}{3}(x^3+1)\,x^2 e^{-x^3}$$

$$e^{-x^3} z = \frac{1}{3}\int x^2 e^{-x^3}dx + \frac{1}{3}\int x^3 x^2 e^{-x^3}dx$$

$$= -\frac{1}{3}\frac{e^{-x^3}}{3} + \frac{1}{3}\left(-\frac{x^3 e^{-x^3}}{3} + \int \frac{3}{3}x^2 e^{-x^3}dx\right)$$

$$= -\frac{e^{-x^3}}{9} - \frac{x^3}{9}e^{-x^3} - \frac{1}{9}e^{-x^3} + c = c - \frac{e^{-x^3}}{9}\left(2 + x^3\right)$$

$$\rightarrow z = ce^{x^3} - \frac{2+x^3}{9}$$

$$y = z^3 = \left(ce^{x^3} - \frac{2+x^3}{9}\right)^3 \quad x \in \mathbb{R}$$

$$y = 0 \quad x \in \mathbb{R}$$

Počáteční podmínka: $y(0)=0$

- triviální řešení splňuje IC: $y = 0 \quad x \in \mathbb{R}$
- netriviální: $c = \frac{2}{9}$

$$\rightarrow y = \left(\frac{2e^{x^3} - 2 + x^3}{9}\right)^3 \quad x \in \mathbb{R}$$

- Pro další c až dva uzlové body