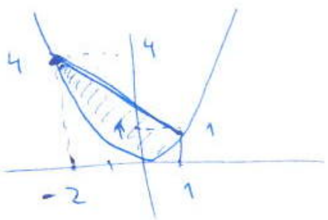
Pokrový sada příkladů 2/2

Spočítejte obsah části roviny omezené následujícími křivkami:

1. $y = x^2 \qquad x + y = 2$

Průsečík: $x^2 = 2 - x \ \dots\ x^2 + x - 2 = (x+2)(x-1) = 0$



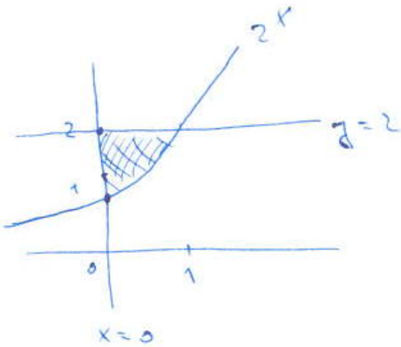
$$P = \int_{-2}^{1} (2-x)\,dx - \int_{-2}^{1} x^2\,dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

4.5

$$= 2 - \frac{1}{2} - \frac{1}{3} - (-4) + 2 + \left(-\frac{8}{3}\right) = 8 - \frac{1}{2} - 3 = \underline{4\tfrac{1}{2}}$$

2. $y = 2^x, \ y = 2, \ x = 0$

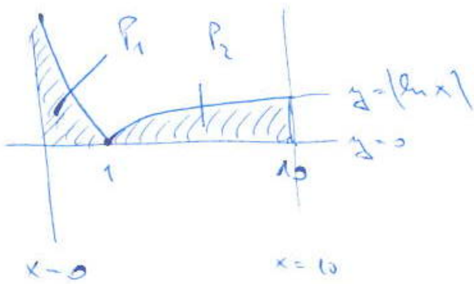


$$P = \int_0^1 2\,dx - \int_0^1 2^x\,dx = 2 - \int_0^1 e^{x\ln 2}\,dx$$

$$= 2 - \left[\frac{e^{x\ln 2}}{\ln 2}\right]_0^1 = 2 - \left[\frac{2^x}{\ln 2}\right]_0^1 = 2 - \frac{2-1}{\ln 2}$$

$$= 2 - \frac{1}{\ln 2}$$

3. $y = |\ln x|, \ y = 0, \ x = 0, \ x = 10$ Reminder: $\int \ln x\,dx = x(\ln x - 1) + c$



$$P = P_1 + P_2$$

$$P_1 = \int_0^1 |\ln x|\,dx = -\int_0^1 \ln x = -\Big[x(\ln x - 1)\Big]_0^1 = 1$$

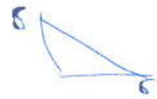
Also $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} \overset{l'H}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$

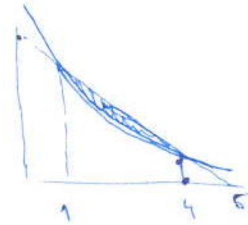
$$= \lim_{x\to 0^+} (-x) = 0 \ \checkmark$$

$$P_2 = \int_1^{10} |\ln x|\,dx = \int_1^{10} \ln x\,dx = \Big[x(\ln x - 1)\Big]_1^{10} = 10\ln 10 - 10 - (0-1) = 10\ln 10 - 9$$

$$\Rightarrow P = P_1 + P_2 = \underline{10(\ln 10 - 1)}$$

4) $xy = 4$, $x+y=5$ .... $y = \frac{4}{x}$, $y = 5-x$



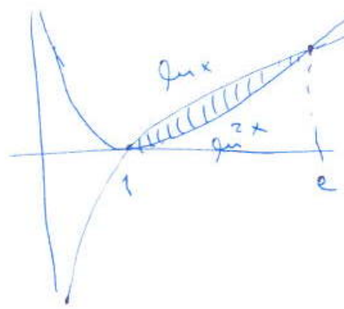


Průsečíky ... 1 a 4

$$P = P_a - P_d = \int_1^4 (5-x)\,dx - \int_1^4 \frac{4}{x}\,dx = \int_1^4 \left(5-x-\frac{4}{x}\right)dx$$

$$= \left[5x - \frac{x^2}{2} - 4\ln x\right]_1^4 = 20 - 8 - 4\ln 4 - 5 + \frac{1}{2} = 7\tfrac{1}{2} - \underbrace{4\ln 4}_{\doteq 5.5452}$$

5) $y=\ln x$, $y=\ln^2 x$ ... $\ln x = \ln^2 x$ ... $x=1 \vee x=e$



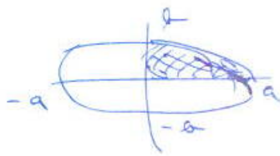
$$\int \ln x\,dx = x(\ln x - 1)$$

$$\int \ln^2 x\,dx = \left|\begin{matrix} u'=1 & u=x \\ v=\ln^2 x & v' = \frac{2\ln x}{x}\end{matrix}\right| = x\ln^2 x - \int 2\ln x\,dx$$
$$= x\ln^2 x - 2x(\ln x - 1)$$

$$P = \int_1^e (\ln x - \ln^2 x)\,dx = \left[x(\ln x - 1) - x\ln^2 x + 2x(\ln x - 1)\right]_1^e$$

$$= \left[3x(\ln x - 1) - x\ln^2 x\right]_1^e = \cancel{3e-3e} - e + 3 = \underline{3-e}$$

6) Obsah elipsy s poloosami a, b ... $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) \rightarrow y = b\sqrt{1-\frac{x^2}{a^2}}$$

$$P = 4\cdot\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \left|\begin{matrix} u = \frac{x}{a} \\ du = \frac{dx}{a}\end{matrix}\right| = 4\cdot ab\underbrace{\int_0^1 \sqrt{1-u^2}}_{\frac14 \text{ unit circle } = \frac{\pi}{4}} = \pi ab$$

$$\int_0^1 \sqrt{1-u^2}\,du = \left|\begin{matrix} u = \sin t \\ du = \cos t\,dt\end{matrix}\right| = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\,dt = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt$$

$$= (\text{[illegible] vzorec z minula}) = \left[\cancel{\frac{\sin t\cos t}{2}}\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\pi/2} dt = \frac{1}{2}\left[t\right]_0^{\pi/2} = \frac{\pi}{4}$$

$$\int \cos^2 t\,dt = \frac{1}{2}(\sin t\cos t + t) + c$$

$$\int \cos^2 t\,dt = \int\left(\frac{1}{2}\cos(2t) + \frac{1}{2}\right)dt = \frac{1}{4}\int \cos(2t)\,2\,dt + \frac{1}{2}\int dt = \frac{\sin 2t}{4} + \frac{t}{2}$$

$$= \frac{1}{2}\left(t + \sin t\cos t\right) \qquad \frac{1}{2}\left[t + \cancel{\sin t\cos t}\right]_0^{\pi/2} = \frac{\pi}{4}$$

(7) Obsah oblasti ohraničené kardioidou $r = a(1+\cos\varphi)$ $\quad a>0$, $\varphi \in [0,2\pi]$

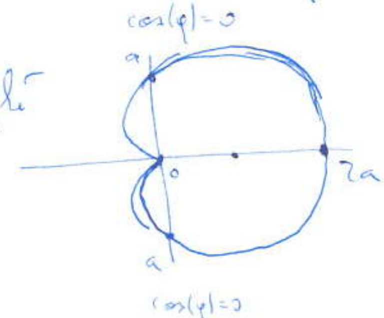
$y = f(x)$ ... parametricky

$x = \varphi(t)$ ... φ monotónní, φ' spojitá

$y = \psi(t)$ ... ψ nezáporná

na $[t_1,t_2]$

$\longrightarrow P = \int_{t_1}^{t_2} \psi(t)\,|\varphi'(t)|\,dt$



• Spec: $\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ \theta \in [\theta_1,\theta_2] \\ 0 < r < f(\theta) \end{array}\right\} \longrightarrow P = \frac{1}{2}\int_{\theta_1}^{\theta_2} f^2(\theta)\,d\theta$

$\downarrow$

$x(\varphi) = a(1+\cos\varphi)\cos\varphi$

$y(\varphi) = a(1+\cos\varphi)\sin\varphi$

$r \equiv f(\varphi) = a(1+\cos\varphi)$

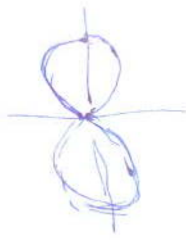
$P_{1/2} = \frac{1}{2}\int_0^\pi f^2(\varphi)\,d\varphi = \frac{1}{2}\int_0^\pi a^2(1+\cos\varphi)^2\,d\varphi = \frac{a^2}{2}\int_0^\pi (1 + 2\cos\varphi + \cos^2\varphi)\,d\varphi$

půl kardioidy

$= \frac{a^2}{2}\int_0^\pi \left[1 + 2\cos\varphi + \frac{1}{2}(1+\cos 2\varphi)\right] d\varphi = \frac{a^2}{2}\int_0^\pi \left(\frac{3}{2} + 2\cos\varphi + \frac{1}{2}\cos 2\varphi\right) d\varphi$

$= \frac{a^2}{2}\left[\frac{3}{2}\varphi + \cancel{2\sin\varphi} + \cancel{\frac{1}{4}\sin(2\varphi)}\right]_0^\pi = \frac{3}{4}a^2\pi$

Celá kardioida: $P = 2\cdot P_{1/2} = \frac{3}{2}\pi a^2$



(8) Obsah oblasti ohraničené lemniskátou $r = 4\sin^2\varphi$, $0 \le \varphi \le 2\pi$

$P = \frac{1}{2}\int_0^{2\pi} r^2\,d\varphi = \frac{1}{2}\int_0^{2\pi} 16\sin^4\varphi\,d\varphi = 8\int_0^{2\pi}\left(\frac{1-\cos 2\varphi}{2}\right)^2 d\varphi$

$\cos(2\varphi) = \cos^2\varphi - \sin^2\varphi = 2\cos^2\varphi - 1 = 1 - 2\sin^2\varphi \Rightarrow$ $\cos^2\varphi = \frac{1+\cos 2\varphi}{2}$, $\sin^2\varphi = \frac{1-\cos 2\varphi}{2}$

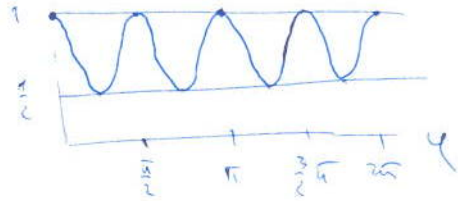
$= 2\int_0^{2\pi}\Big(1 - 2\cos(2\varphi) + \underbrace{\cos^2(2\varphi)}_{\frac{1+\cos 4\varphi}{2}}\Big)\,d\varphi = 2\left[\varphi - \cancel{\sin(2\varphi)} + \frac{\varphi}{2} + \cancel{\frac{\sin 4\varphi}{8}}\right]_0^{2\pi} = 2\cdot\frac{3}{2}\cdot 2\pi$

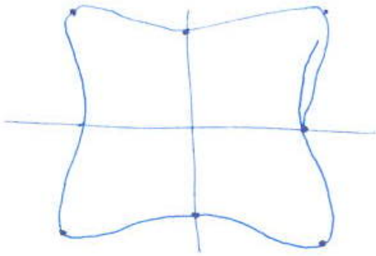
$= 6\pi$

9 Obsah oblasti ohraničené $x^4+y^4 = x^2+y^2$ ... ?

Zavedeme: $x = r\cos\varphi$, $y = r\sin\varphi$ $\rightarrow$ $r^4(\cos^4\varphi+\sin^4\varphi) = r^2\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}$

$$r^2 = \frac{1}{\cos^4\varphi+\sin^4\varphi}$$





$$P = \frac{1}{2}\int_0^{2\pi} \frac{1}{\cos^4\varphi+\sin^4\varphi}\,d\varphi = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos^4\varphi+\sin^4\varphi}\,d\varphi = 2\cdot\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{d\varphi}{\cos^4\varphi+\sin^4\varphi}$$

změna intervalu ... $\frac{\pi}{2}$-periodická fce OK

$$\cos^4\varphi+\sin^4\varphi = \left(\frac{1+\cos 2\varphi}{2}\right)^2 + \left(\frac{1-\cos 2\varphi}{2}\right)^2 = \frac{1}{4}+\cos 2\varphi+\frac{\cos^2 2\varphi}{4}+\frac{1}{4}-\cos 2\varphi+\frac{\cos^2 2\varphi}{4}$$
$$= \frac{1+\cos^2 2\varphi}{2}$$

$$\rightarrow 2\cdot\int \frac{2}{1+\cos^2(2\varphi)}\,d\varphi = \left|\begin{matrix} 2\varphi = t \\ 2\,d\varphi = dt\end{matrix}\right| = 2\int \frac{dt}{1+\cos^2 t} = \left|\begin{matrix} v = \tan t \\ dv = \frac{dt}{\cos^2 t} \\ \frac{dv}{1+v^2} = dt\end{matrix}\right|$$

$$= 2\int \frac{1}{1+\frac{1}{1+v^2}}\,\frac{dv}{1+v^2} = 2\int\frac{dv}{v^2+2} = \int \frac{dv}{\left(\frac{v}{\sqrt{2}}\right)^2+1} \quad \left|\begin{matrix}\frac{v}{\sqrt{2}} = u \\ \frac{dv}{\sqrt{2}} = du\end{matrix}\right.$$

$$= \int \frac{\sqrt{2}\,du}{u^2+1} = \sqrt{2}\int\frac{du}{u^2+1} = \sqrt{2}\arctan u = \sqrt{2}\arctan\left(\frac{v}{\sqrt{2}}\right)$$

$$= \sqrt{2}\arctan\left(\frac{\tan t}{\sqrt{2}}\right) = \sqrt{2}\arctan\left(\frac{\tan(2\varphi)}{\sqrt{2}}\right)$$

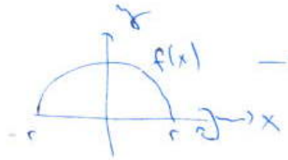
$$P = \left[\sqrt{2}\arctan\left(\frac{\tan(2\varphi)}{\sqrt{2}}\right)\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \sqrt{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \sqrt{2}\,\pi$$

ve smyslu lim

(10) Vztah pro objem koule, kužele, jehlanu

Koule ... rotační těleso, rotace funkce $f(x)=y$ kolem osy x na intervalu $[a,b]$ $(f(x)\geq 0$ na $a,b)$ $\rightarrow$ $V=\pi\int_a^b f^2(x)\,dx$

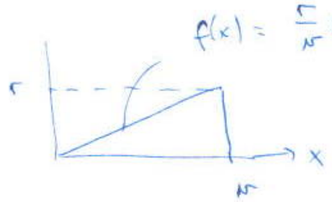
$y=f(x)=\sqrt{r^2-x^2}\quad x\in[-r,r]$



$\rightarrow V=\pi\int_{-r}^{r}(r^2-x^2)\,dx=\pi\left[r^2x-\frac{x^3}{3}\right]_{-r}^{r}=\pi\underbrace{\left(r^3+r^3-\frac{r^3}{3}-\frac{r^3}{3}\right)}_{\left(2-\frac{2}{3}\right)r^3=\frac{4}{3}r^3}=\frac{4}{3}\pi r^3$

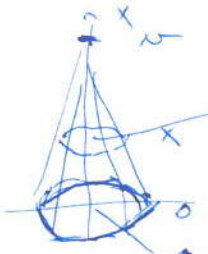
Kužel výška v a poloměr podstavy r

$f(x)=\frac{r}{v}x\quad x\in[0,v]$



$V=\pi\int_0^v\frac{r^2}{v^2}x^2\,dx=\pi\frac{r^2}{v^2}\left[\frac{x^3}{3}\right]_0^v=\pi\frac{r^2}{v^2}\frac{v^3}{3}=\frac{1}{3}\pi r^2 v$

Jehlan $\quad V=\int_0^v S(x)\,dx=S\int_0^v\left(1-\frac{x}{v}\right)^2dx=S\int_0^v\left(1-\frac{2x}{v}+\frac{x^2}{v^2}\right)dx$



délky hran lineárně klesají s výškou v poměru $\frac{v-x}{v}=1-\frac{x}{v}$

$\rightarrow$ plocha průřezu $S(x)$ se mění jako $\left(1-\frac{x}{v}\right)^2$

$S=S(0)$

$=S\left[x-\frac{x^2}{v}+\frac{x^3}{3v^2}\right]_0^v=\left(\cancel{v}-\cancel{v}+\frac{v^3}{3v^2}\right)S=\frac{1}{3}Sv$

(11) Objem tělesa vzniklého rotací kardioidy $r=a(1+\cos\varphi)$ $\varphi\in[0,\pi]$ kolem osy x

$\rightarrow$ Rotace křivky $r=f(\varphi)$ $\varphi\in[\alpha,\beta]\subseteq[0,\pi]$

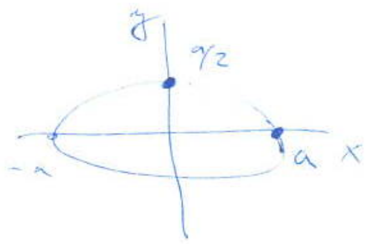
$V=\frac{2}{3}\pi\int_\alpha^\beta f^3(\varphi)\sin\varphi\,d\varphi \Rightarrow V=\frac{2}{3}\pi\int_0^\pi a^3(1+\cos\varphi)^3\sin\varphi\,d\varphi$

$=\left|\begin{array}{l}1+\cos\varphi=u\\ -\sin\varphi\,d\varphi=du\\ u\text{ from 2 to 0}\end{array}\right|=-\frac{2}{3}\pi\int_2^0 a^3u^3\,du=+\frac{2}{3}\pi a^3\int_0^2u^3\,du$

$=\frac{2}{3}\pi a^3\left[\frac{u^4}{4}\right]_0^2=\frac{2}{3}\pi a^3\frac{16}{4}=\frac{8}{3}\pi a^3$

(12) Objem části tělesa $\underbrace{x^2+4y^2 \le a^2}$ ležícího mezi rovinami $z=0$ a $y=z$

elipsa → nekonečný eliptický válec; $z=0$: odsekne zdola; $y=z$: zhora



Integrace podél osy y přes obdélníky o svislé hraně $z=y$ a vodorovné hraně $x = 2\sqrt{a^2-4y^2}$

$$V=\int_0^{a/2} S(y)\,dy = \int_0^{a/2} 2y\sqrt{a^2-4y^2}\,dy = \left|\begin{array}{l} a^2-4y^2=u \\ -8y\,dy = du \end{array}\right| = -\frac{1}{4}\int_{a^2}^{0}\sqrt{u}\,du$$

u from a^2 do 0

$$= +\frac{1}{4}\int_0^{a^2}\sqrt{u}\,du = \frac{1}{4}\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_0^{a^2} = \frac{1}{4}\,\frac{2}{3}\,a^3 = \frac{a^3}{6}$$

(13) Délka kružnice ... slíbil v počátku

Metoda 1: Křivka popsaná grafem na (a,b) se spojitou první derivací platí

$$l=\int_a^b\sqrt{1+(f'(x))^2}\,dx$$

→ Půlkružnice: $f(x)=\sqrt{r^2-x^2}$ → $f'=-\frac{x}{\sqrt{r^2-x^2}}$

$$l=\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx = \int_{-r}^{r}\frac{r}{\sqrt{r^2-x^2}} = \left|\begin{array}{l} x=r\sin t \\ dx = r\cos t\,dt \\ t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \end{array}\right| = \int_{-\pi/2}^{\pi/2}\frac{r}{\sqrt{r^2-r^2\sin^2 t}}\,r\cos t\,dt$$

$$= \int_{-\pi/2}^{\pi/2}\frac{r}{r\cos t}\,r\cos t\,dt = r\int_{-\pi/2}^{\pi/2}dt = \pi r \quad \to \text{ celá kružnice } = 2\pi r$$

Metoda 2: Křivka popsaná parametricky $x=\varphi(t)$, $y=\psi(t)$ $t\in[t_1,t_2]$ za předpokladu, že φ' a ψ' jsou stejnoměrně spojité na (t_1,t_2) $\Rightarrow l=\int_{t_1}^{t_2}\sqrt{(\varphi'(t))^2+(\psi'(t))^2}\,dt$

→ Kružnice: $x=r\cos t=\varphi(t)$ → $\varphi'=-r\sin t$

$y=r\sin t=\psi(t)$ → $\psi'=r\cos t$

$t\in[0,2\pi)$

$$l=\int_0^{2\pi}\sqrt{(-r\sin t)^2+(r\cos t)^2}\,dt = \int_0^{2\pi} r\,dt = 2\pi r$$

Metoda 3: Křivka určená rovnicí $r = f(\varphi)$ $\varphi \in [\varphi_1, \varphi_2]$ v polárních souřadnicích, za předpokladu stejnoměrné spojitosti f' na (φ_1, φ_2), platí

$$l = \int_{\varphi_1}^{\varphi_2} \sqrt{(f(\varphi))^2 + (f'(\varphi))^2}\, d\varphi$$

Kružnice: $r = R \equiv f(\varphi)$, $\varphi \in [0, 2\pi]$ $\rightarrow$ $l = \int_0^{2\pi} \sqrt{R^2 + 0^2}\, d\varphi = 2\pi R$

(14) Spočtěte délku křivky $y = \arcsin x + \sqrt{1-x^2}$, $x \in (-1, 1)$

$$y = f(x) = \dots \qquad f' = \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \frac{1-x}{\sqrt{1-x^2}} \qquad (f')^2 = \frac{(1-x)^2}{1-x^2} = \frac{1-x}{1+x}$$

$$\rightarrow l = \int_{-1}^{1} \sqrt{1 + (f'(x))^2}\, dx = \int_{-1}^{1} \sqrt{1 + \frac{1-x}{1+x}}\, dx = \int_{-1}^{1} \sqrt{\frac{2}{1+x}}\, dx = \left[2\sqrt{2}\sqrt{1+x}\right]_{-1}^{1}$$

$$= 2\sqrt{2}\left(\sqrt{2} - 0\right) = \underline{\underline{4}}$$

(15) Spočtěte délku evolventy kruhu $x = a(\cos t + t\sin t)$ $y = a(\sin t - t\cos t)$ $t \in [0, 2\pi]$

$$\rightarrow l = \int_0^{2\pi} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{2\pi} \sqrt{\left(a(-\cancel{\sin t} + \cancel{\sin t} + t\cos t)\right)^2 + \left(a(\cancel{\cos t} - \cancel{\cos t} + t\sin t)\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(at\cos t)^2 + (at\sin t)^2}\, dt = \int_0^{2\pi} \sqrt{a^2 t^2}\, dt = \int_0^{2\pi} at\, dt$$

$$= \left[\frac{at^2}{2}\right]_0^{2\pi} = \frac{a}{2} 4\pi^2 = \underline{2\pi^2 a}$$

(16) Odvoďte vzorec pro povrch koule.

Metoda 1: Obsah plochy, jež vznikne rotací křivky, jež je grafem funkce f na $[a,b]$, je-li f' stejnoměrně spojitá na (a,b), vypočteme podle vztahu

$$S = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2}\, dx$$

→ rotace půlkružnice: $f(x) = \sqrt{R^2 - x^2}$ $\quad x \in [-R,R]$ kolem osy x

$$|f(x)| = f(x) \qquad f'(x) = -\frac{x}{\sqrt{R^2-x^2}}$$

$$P = 2\pi \int_{-R}^{R} \sqrt{R^2-x^2} \sqrt{1+\frac{x^2}{R^2-x^2}}\, dx = 2\pi \int_{-R}^{R} \cancel{\sqrt{R^2-x^2}} \sqrt{\frac{R^2-\cancel{x^2}+\cancel{x^2}}{\cancel{R^2-x^2}}} = 2\pi \int_{-R}^{R} R\, dx$$

$$= 2\pi R\, [x]_{-R}^{R} = \underline{4\pi R^2}$$

Metoda 2: Křivka popsaná parametricky $x = \varphi(t)$, $y = \psi(t)$, $t \in (\alpha, \beta)$, φ' a ψ' stejnoměrně spojité a φ ryze monotónní → $S = 2\pi \int_\alpha^\beta |\psi(t)| \sqrt{(\varphi'(t))^2 + (\psi'(t))^2}$

$x = R\cos t = \varphi(t)$ $\quad t \in [0, 2\pi]$ $\quad \varphi' = -R\sin t$

$y = R\sin t = \psi(t)$ $\quad$ půlkružnice $\quad \psi' = R\cos t$

$$\to P = 2\pi \int_0^\pi R\sin t \underbrace{\sqrt{R^2\sin^2 t + R^2\cos^2 t}}_{R^2} = 2\pi R^2 \int_0^\pi \sin t\, dt = 2\pi R^2 \underbrace{[-\cos t]_0^\pi}_{2} = 4\pi R^2$$

Metoda 3: Rotace křivky $r = f(\varphi)$ $\varphi \in [\varphi_1, \varphi_2] \subset [0,\pi]$ kolem polární osy, f' stejnoměrně spojitá na (φ_1, φ_2) → $S = 2\pi \int_{\varphi_1}^{\varphi_2} |f(x)| \sin\varphi \sqrt{(f(\varphi))^2 + (f'(\varphi))^2}\, d\varphi$

→ $\underset{f(\varphi)}{\overset{}{r = R}}$, $\varphi \in [0,\pi]$ $\qquad P = 2\pi \int_0^\pi R\sin\varphi \sqrt{R^2 + 0^2}\, d\varphi = 2\pi R^2 \int_0^\pi \sin\varphi = 2\pi R^2 [-\cos\varphi]_0^\pi$
$= \underline{4\pi R^2}$

(17) Povrch rotačního tělesa, rotace křivky $y = x^3$, $|x| \le 1$ kolem osy x

→ zadáno grafem funkce: $y = x^3 = f(x)$ $\quad x \in [-1, 1]$

$$P = 2\pi \int_{-1}^{1} |f(x)| \sqrt{1+(f'(x))^2}\, dx = 2\pi \int_{-1}^{1} |x^3| \sqrt{1+9x^4}\, dx \underset{\text{symetrie + sudost}}{=} 4\pi \int_0^1 x^3 \sqrt{1+9x^4}\, dx =$$

$$= \left| \begin{matrix} 1+9x^4 = u \\ 36x^3 dx = du \\ u \to 1 .. 10 \end{matrix} \right| = \frac{4\pi}{36} \int_1^{10} \sqrt{u}\, du = \frac{\pi}{9} \left[\frac{u^{3/2}}{3/2}\right]_1^{10} = \frac{2\pi}{27} \left[10^{3/2} - 1\right]$$

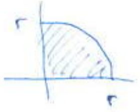
(18) Nalezněte těžiště homogenního čtvrtkruhu o poloměru r.

- Oblast $a \le x \le b$, $g(x) \le y \le f(x)$, f, g spojité, $\sigma(x)$ plošná hustota
- statické momenty vzhledem k ose x, y:

$$M_x = \frac{1}{2}\int_a^b \sigma(x)\left(f^2(x) - g^2(x)\right)dx \qquad \text{Těžiště: } T = [\xi, \eta], \text{ kde } \xi = \frac{M_y}{M} \quad \eta = \frac{M_x}{M}$$

$$M_y = \int_a^b \sigma(x)\, x \left(f(x) - g(x)\right)dx \qquad \text{kde } M = \int_a^b \sigma(x)\left(f(x) - g(x)\right)dx$$

→ čtvrtkruh: $f(x) = \sqrt{r^2 - x^2}$, $g(x) = 0$, $x \in [0, r]$, $\sigma(x) = \text{const}\ (=1)$



$$M = \int_0^r \sqrt{r^2 - x^2}\,dx = \left| \begin{array}{l} x = r\sin t \\ dx = r\cos t\,dt \\ t = 0 \dots \frac{\pi}{2} \end{array} \right| = \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2 t}\,\cos t\,dt$$

$$= r^2 \underbrace{\int_0^{\pi/2} \cos^2 t\,dt}_{\frac{\pi}{4}} = \frac{\pi}{4} r^2$$

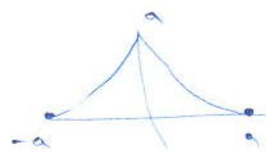
$$M_x = \frac{1}{2}\int_0^r (r^2 - x^2)\,dx = \frac{1}{2}\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{1}{2}\cdot\frac{2}{3} r^3 = \frac{1}{3} r^3$$

$$M_y = \int_0^r x\sqrt{r^2 - x^2}\,dx = \left| \begin{array}{l} r^2 - x^2 = u \\ -2x\,dx = du \\ u = r^2 \dots 0 \end{array} \right| = \int_{r^2}^0 \sqrt{u}\left(-\frac{du}{2}\right) = \frac{1}{2}\int_0^{r^2} \sqrt{u}\,du = \frac{1}{2}\,\frac{u^{3/2}}{\frac{3}{2}}$$

$$= \frac{1}{2}\cdot\frac{2}{3} r^3 = \frac{1}{3} r^3 \qquad \text{Těžiště: } T = [\xi, \eta] \qquad \xi = \frac{M_y}{M} = \frac{\frac{r^3}{3}}{\frac{\pi}{4} r^2} = \frac{4r}{3\pi}$$

$$\eta = \frac{M_x}{M} = \frac{4r}{3\pi} \approx 0.42\,r$$

(19) Nalezněte polohu těžiště poloviny homogenní asteroidy $x = a\cos^3 t$, $y = a\sin^3 t$, $t \in [0, \pi]$



- křivka/oblouk $x = \varphi(t)$, $y = \psi(t)$ s lineární hustotou $\mu(t)$, $t \in [t_1, t_2]$, φ', ψ' spojité

$$M_x = \int_{t_1}^{t_2} \mu(t)\psi(t)\sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\,dt$$

$$M_y = \int_{t_1}^{t_2} \mu(t)\varphi(t)\sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\,dt$$

$$T = [\xi, \eta] \quad \xi = \frac{M_y}{M} \quad \eta = \frac{M_x}{M} \quad M = \int_{t_1}^{t_2} \mu(t)\sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\,dt$$

⋮

asteroida next page

19 cont'd

$x = \varphi(t) = a\cos^3 t \qquad \varphi' = -3a\cos^2 t \sin t$

$y = \psi(t) = a\sin^3 t \qquad \psi' = 3a\sin^2 t \cos t$

$t \in [0, \pi]$

$\mu(t) = \text{const} \ (=1)$

$$M = \int_{t_1}^{t_2} \mu(t)\sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\,dt = \int_0^{\pi} \sqrt{9a^2\cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t}\,dt$$

$$= 3a \int_0^{\pi} |\cos t \sin t|\,dt = 3a\int_0^{\pi/2} \sin t \cos t\,dt - 3a \int_{\pi/2}^{\pi} \sin t \cos t\,dt$$

mění znaménko v $\frac{\pi}{2}$

$$= \frac{3a}{2}\int_0^{\pi/2} \sin(2t)\,dt - \frac{3a}{2}\int_{\pi/2}^{\pi} \sin(2t)\,dt = \frac{3a}{2}\underbrace{\left[-\frac{\cos 2t}{2}\right]_0^{\pi/2}}_{+1} - \frac{3a}{2}\underbrace{\left[-\frac{\cos 2t}{2}\right]_{\pi/2}^{\pi}}_{+1} = 3a$$

$$M_x = \int_{t_1}^{t_2} \mu(t)\psi(t)\sqrt{(\varphi'(t))^2 + (\psi'(t))^2}\,dt = \int_0^{\pi} a\sin^3 t \sqrt{9a^2 \ldots}\,dt$$

$$= 3a^2 \int_0^{\pi} \sin^3 t\,|\sin t \cos t| = 3a^2 \int_0^{\pi/2} \sin^4 t \cos t\,dt - 3a^2 \int_{\pi/2}^{\pi} \sin^4 t \cos t\,dt$$

$$= \left| \begin{matrix} \sin t = u \\ \cos t\,dt = du \end{matrix} \right| = 3a^2 \int_0^1 u^4\,du - 3a^2 \int_1^0 u^4\,du = 6a^2 \int_0^1 u^4\,du$$

$$= 6a^2 \left[\frac{u^5}{5}\right]_0^1 = \frac{6}{5}a^2$$

$T = [\xi, \eta] \qquad \xi = \frac{M_y}{M} = 0$

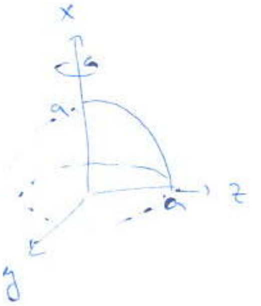
$\eta = \frac{M_x}{M} = \frac{\frac{6}{5}a^2}{3a} = \frac{2}{5}a$

$M_y = \ldots\ \emptyset$ (ze symetrie)

(20) Nalezněte polohu těžiště homogenní polokoule $x^2+y^2+z^2 \le a^2$, $x>0$.

- Rotační těleso $a \le x \le b$, $0 \le g(x) \le \sqrt{y^2+z^2} \le f(x)$ s objemovou hustotou $\rho(x)$
- Statické momenty vzhledem k souřadnicovým rovinám

$$M_{xy} = M_{xz} = 0 \qquad M_{yz} = \pi \int_a^b \rho(x)\, x \left(f^2(x)-g^2(x)\right) dx$$



Těžiště $T=(\xi,0,0)$ $\quad \xi = \frac{M_{yz}}{M} \quad M = \pi\int_a^b \rho(x)\left(f^2(x)-g^2(x)\right) dx$

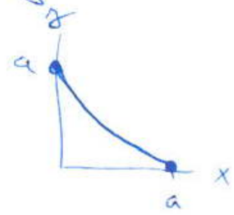
→ Rotace čtvrtkruhu $x\in[0,a]$ $\quad f(x)=\sqrt{a^2-x^2} \quad g(x)=0 \quad \rho(x)=\text{konst}$ $(=1)$

$$M = \pi\int_0^a (a^2-x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{2}{3}\pi a^3 \ \dots \ \tfrac{1}{2}\text{ koule} \checkmark$$

$$M_{yz} = \pi\int_0^a x(a^2-x^2)\,dx = \pi\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a = \pi\left(\frac{a^4}{2}-\frac{a^4}{4}\right) = \pi\frac{a^4}{4}$$

$$T=(\xi,0,0) \qquad \xi = \frac{M_{yz}}{M} = \frac{\frac{\pi a^4}{4}}{\frac{2}{3}\pi a^3} = \frac{3}{8}a$$

(21) Moment setrvačnosti oblouku asteroidy vzhledem k souřadnicovým osám
$x=a\cos^3 t$, $y=a\sin^3 t$, $t\in[0,\frac{\pi}{2}]$



statický moment $\int r\,dm$ ... r ... kolmá vzdálenost dm vůči ose

moment setrvačnosti $\int r^2\,dm$

$$I_x = \int_{t_1}^{t_2} \mu(t)\,\psi^2(t)\sqrt{(\varphi'(t))^2+(\psi'(t))^2}\,dt \qquad \begin{aligned} x&=\varphi(t)=a\cos^3 t\\ y&=\psi(x)=a\sin^3 t\\ \mu(t)&=\text{konst } (=1)\end{aligned}$$

$$I_y = \int_{t_1}^{t_2} \mu(t)\,\varphi^2(t)\sqrt{\qquad}\,dt$$

$$\to I_x = \int_0^{\pi/2} a^2\sin^6 t\sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} = 3a^3\int_0^{\pi/2} \sin^7 t\cos t\,dt$$

$$= \left| \begin{aligned} u&=\sin t\\ du&=\cos t\,dt\\ u&\in[0,1]\end{aligned} \right| = 3a^3\int_0^1 u^7\,du = 3a^3\left[\frac{u^8}{8}\right]_0^1 = \underline{\frac{3}{8}a^3}$$

$I_y = I_x$ (symetrie)

(22) Přímočarý pohyb tělesa je dán funkcí $s = ct^3$, kde $s(t)$ je délka dráhy za čas t. Odpor prostředí je úměrný čtverci rychlosti. Vypočítejte práci, kterou vykonají třecí síly, pokud těleso projde dráhu od $s=0$ do $s=a$.

- rychlost: $v = \frac{ds}{dt} = 3ct^2$
- Třecí síla: $F = kv^2 = 9kc^2t^4$
- Pohyb od $s=0$ do $s=a$ $\rightarrow$ $t=0$ do $\sqrt[3]{\frac{a}{c}}$
- Práce F síl: $W = \int \vec{F}\cdot d\vec{s} = \int_0^a F\,ds = \int_{t_1=0}^{t_2=\sqrt[3]{\frac{a}{c}}} 9kc^2t^4\, 3ct^2\,dt$

$$= 27kc^3 \int_0^{\sqrt[3]{\frac{a}{c}}} t^6\,dt = 27kc^3\left[\frac{t^7}{7}\right]_0^{\sqrt[3]{\frac{a}{c}}} = \frac{27}{7}kc^3\sqrt[3]{\frac{a^7}{c^7}}$$

$$= \underline{\frac{27}{7}k\sqrt[3]{a^7c^2}}$$

(23) Při průchodu radioaktivního záření vrstvou látky o tloušťce h poklesla jeho intenzita na polovinu původní hodnoty. Jaká bude intenzita tohoto záření po průchodu vrstvou o tloušťce H? (Předpoklad: intenzita záření absorbovaného tenkou vrstvou látky je přímo úměrná tloušťce vrstvy a intenzitě dopadajícího záření)

- I_0 ... dopadající
- vrstva o tloušťce dx: $dI = -kI\,dx$ ... $I'(x) = -kI$

$\rightarrow I(x) = I_0 e^{-kx}$

$$\frac{dI}{I} = -k\,dx \qquad x = h \Rightarrow \frac{I}{I_0} = \frac{1}{2}$$

$$\frac{I}{I_0} = \frac{1}{2} = e^{-kh}$$

$$\ln\frac{1}{2} = -kh$$

$$\ln 2 = kh$$

$$k = \frac{\ln 2}{h}$$

$$I = I_0 e^{-\frac{x}{h}\ln 2}$$

$$H: \quad I_0 - I(H) = I_0 - I_0 e^{-\frac{H}{h}\ln 2} = I_0\left[1 - \left(\frac{1}{2}\right)^{\frac{H}{h}}\right]$$