Matematický formalizmus

Krivočiare súradnice

- vektory $\vec{a} = a^i \frac{\partial}{\partial x^i} \in \mathcal{T}\mathcal{M}$ tečný prostor variety
- kovektory $\omega = \omega_i \, dx^i \in \mathcal{T}^*\mathcal{M}$ kotečný prostor variety
- metrika $g_{ij} : \mathcal{T}\mathcal{M} \times \mathcal{T}\mathcal{M} \to \mathbb{R}$ → bilin forma $\in \mathcal{T}^*\mathcal{M} \times \mathcal{T}^*\mathcal{M}$

$$g(a,b) = g_{ij} \cdot a^i b^j = a_j b^j = a^i b_i \quad \to \text{definyje skalarny súčin}$$

$\left\{ \frac{\partial}{\partial x^i} \right\}$ → vektorová báze

$$\frac{\partial}{\partial x^i} \cdot dx^j = \delta_i^j$$

$\left\{ dx^i \right\}$ → kovektorová báze

- tenzor $\mathbf{T} = T^{ij\ldots}_{kl\ldots} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \ldots dx^k \, dx^l$
- ortogonální souřadnice:

$$g = \begin{pmatrix} h_1^2 & & \\ & h_2^2 & \\ & & h_3^2 \end{pmatrix} \qquad h_i \to \text{Lámeho koeficienty}$$

$$\left. \begin{aligned} e^{\hat{a}} &= h_a \, dx^a \\ e_{\hat{b}} &= \frac{1}{h_b} \frac{\partial}{\partial x^b} \end{aligned} \right\} \text{ortonormální báze} \; \hookrightarrow \text{funkce}$$

Kovariantní derivace

$$\nabla\left(a^j \frac{\partial}{\partial x^j}\right) = \nabla a^j \frac{\partial}{\partial x^j} + a^j \underbrace{\nabla \frac{\partial}{\partial x^j}}_{\Gamma^i_{kj} \, dx^k \frac{\partial}{\partial x^i}}$$

Γ^i_{kj} → Christoffelove symboly

$$\nabla(\omega_j \, dx^j) = \nabla \omega_j \, dx^j + \omega_j \underbrace{\nabla dx^j}_{-\Gamma^j_{ik} \, dx^i dx^k}$$

$$\Gamma^k_{lm} = \frac{\partial}{\partial x^l} \cdot \nabla \frac{\partial}{\partial x^m} \cdot dx^k = \frac{\partial}{\partial x^l} \left(\nabla \frac{\partial x^i}{\partial x^m} \frac{\partial}{\partial x^i} \right) \cdot \frac{\partial x^k}{\partial x^n} dx^n =$$

$$= \frac{\partial}{\partial x^l} \cdot \left(\nabla \frac{\partial x^i}{\partial x^m} \right) \frac{\partial x^k}{\partial x^n} \underbrace{\frac{\partial}{\partial x^i} \cdot dx^n}_{\delta_i^n} = \frac{\partial}{\partial x^l} \cdot \left(\nabla \frac{\partial x^i}{\partial x^m} \right) \frac{\partial x^k}{\partial x^i} = \frac{\partial^2 x^i}{\partial x^l \partial x^m} \frac{\partial x^k}{\partial x^i}$$

• Γ pomocou metriky:

$$g_{kl;m}=0 \quad \Rightarrow \quad \Gamma^k_{lm} = \frac{1}{2} g^{ik}\left(g_{im,l} + g_{li,m} - g_{ml,i}\right)$$

• Γ pre OG súradnice

$$g_{ij} = h_i\, \delta_{ij}$$

$$\Gamma^k_{lm} = 0 \quad \text{pre} \quad k \neq m \neq l$$

$$\Gamma^k_{lm} = \frac{1}{2}\frac{1}{h_i^2}\delta^{ik}\left(h^2_{i,l}\,\delta_{im} + h^2_{i,m}\,\delta_{li} - h^2_{m,i}\,\delta_{ml}\right) =$$

$$= \frac{h_{m,l}}{h_m}\delta^k_m + \frac{h_{l,m}}{h_l}\delta^k_l - \frac{h_m h_m^{,k}}{h_k^2}\delta_{ml}$$

$$\Rightarrow \sum_k \Gamma^k_{km} = \sum \frac{h_{k,m}}{h_k} = \ln(h_1h_2h_3)_{,m} = \frac{(h_1h_2h_3)_{,m}}{h_1h_2h_3}$$

Operátory v krivých súradniciach

• gradient $\quad (\nabla f)_m = \nabla_m f = f_{,m} \quad \rightsquigarrow \quad \boxed{f_{,\hat{m}} = \frac{1}{h_m} f_{,m}}$

• divergence

$$\nabla\cdot\vec{a} = a^k{}_{;k} = a^k{}_{,k} + \Gamma^k_{km} a^m = a^k{}_{,k} + \ln(h_1h_2h_3)_{,k}\, a^k =$$

$$= \frac{(a^k h_1h_2h_3)_{,k}}{h_1h_2h_3}$$

normalizace

$$\boxed{\nabla\cdot\vec{a} = \frac{1}{h_1h_2h_3}\left[(a^1 h_2h_3)_{,1} + (a^2 h_1h_3)_{,2} + (a^3 h_1h_2)_{,3}\right]}$$

• rotace

$$(\nabla\times\vec{a})^k = \varepsilon^{klm} a_{m;l} = \varepsilon^{klm} a_{m,l} - \overset{0}{\cancel{\varepsilon^{\overset{\text{antisym}}{klm}}\,\Gamma^n_{\overset{\text{sym.}}{lm}}\, a_n}} =$$

$$= \varepsilon^{klm} a_{m,l} \qquad \varepsilon^{123} = (h_1h_2h_3)^{-1}$$

$$(\nabla\times\vec{a})^1 = \frac{1}{h_1h_2h_3}\left(a_{3,2} - a_{2,3}\right) = \frac{1}{h_1h_2h_3}\left((h_3^2 a^3)_{,2} - (h_2^2 a^2)_{,3}\right)$$

normalizace

$$\boxed{(\nabla\times\vec{a})^1 = \frac{1}{h_2h_3}\left((h_3 a^3)_{,2} - (h_2 a^2)_{,3}\right) + \text{cyklické záměny}}$$

• Laplacián

$$\boxed{\nabla^2 f = \frac{1}{h_1h_2h_3}\left[\left(\frac{h_2h_3}{h_1} f_{,1}\right)_{,1} + \left(\frac{h_1h_3}{h_2} f_{,2}\right)_{,2} + \left(\frac{h_1h_2}{h_3} f_{,3}\right)_{,3}\right]}$$

Elekrostatika

Formulace elektrostatiky

- žiadne časové závislosti
- iba rozloženie náboja

$$\boxed{\begin{aligned}\nabla\cdot\vec{E} &= \frac{\rho}{\varepsilon_0}\\ \nabla\times\vec{E} &= 0\end{aligned}}$$

$\vec{f} = q\vec{E}$ → sila na náboj

Gaussov zákon

$$\boxed{\oint_{\partial\Omega}\vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}Q}$$

→ siločiary vznikajú na náboji

↳ rovnice siločiary $\dfrac{d\vec{x}(s)}{ds} = \lambda(s)\,\vec{E}(\vec{x}(s))$

Potenciál

$\nabla\times\vec{E} = 0 \iff \vec{E}$ je konzer. $\iff \oint_\Gamma \vec{E}\cdot d\vec{l} = 0 \iff \int_\Gamma \vec{E}\cdot d\vec{l}$ nezávisí na Γ

↳ dle Poincarého ex. ϕ: $\vec{E} = -\nabla\phi$

$$\Rightarrow \int_\Gamma \vec{E}\cdot d\vec{l} = \phi(\Gamma_{end}) - \phi(\Gamma_{start})$$

$\nabla\times\vec{E} \iff \vec{E} = -\nabla\phi$ Poincaré

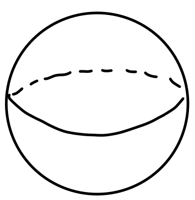
"$\Leftarrow$" $\nabla\times\nabla\phi = \varepsilon_{ijk}\,\partial_j\,\partial_k\,\phi = 0$

"$\Rightarrow$" nelze vždy → jednoduše souvislé oblasti

$\nabla\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \Rightarrow -\nabla\cdot\nabla\phi = \frac{1}{\varepsilon_0}\rho \Rightarrow \boxed{\nabla^2\phi = -\frac{1}{\varepsilon_0}\rho}$ Poisson

$\nabla^2\phi = 0$ Laplace

Coulombov zákon



$\rho = \delta(\vec{x}-\vec{x}')\,Q$

$\vec{E} = E(r)\,\vec{e}_r$ → sféricky symetrický

$$\oint_{koule}\vec{E}\cdot d\vec{S} = 4\pi r^2 E(r) = \frac{1}{\varepsilon_0}Q \Rightarrow \vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{e}_r$$

Singulární zdroje

$\rho \to$ distribuce nábojа

Regulární zdroje (3D)

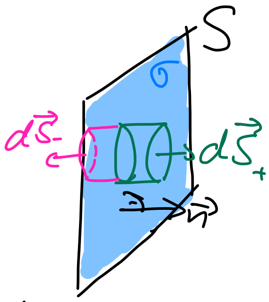
$\to$ objemově rozložený náboj $\quad Q = \int \rho \, dV \quad \to$ nultý moment distribuce

$\to \vec{E}$ je spojité, nesingulárne

Plošné zdroje (2D)

$\to$ náboj rozložený na ploche $\quad Q = \int \sigma \, dS$

$\to$ nespojitost $\vec{E}$ na ploche



$$\rho(x) = \sigma(x)\delta_S(x) = \sigma(x)\, \mathcal{H}^2\big|_S$$

$\chi_\pm \to$ charakteristické funkce stran plochy

$$\vec{E} = \chi_+ \vec{E}_+ + \chi_- \vec{E}_-$$

$$\nabla\cdot\vec{E} = \nabla\cdot\vec{E}_+ \chi_+ + \nabla\cdot\vec{E}\chi_- + \vec{E}_+\cdot\nabla(\chi_+) + \vec{E}_-\cdot\nabla(\chi_-) =$$

$$\left[\int \nabla\chi_\Omega\cdot\vec{\psi}\, dV := \int -\chi_\Omega \nabla\cdot\vec{\psi}\, dV = -\int_\Omega \nabla\cdot\vec{\psi}\, dV = \int_{\partial\Omega} -\vec{\psi}\cdot d\vec{S} = \right.$$
$$\left. = \int \vec{\psi}\cdot\vec{n}\,\delta_{\partial\Omega}\, dV \right]$$

$$= \nabla\cdot\vec{E}_+\chi_+ + \nabla\cdot\vec{E}\chi_- + \vec{E}_+\cdot\vec{n}_+\delta_S + \vec{E}_-\cdot\vec{n}_-\delta_S = \frac{1}{\varepsilon_0}\sigma\delta_S$$

$$\Rightarrow \boxed{[\vec{E}]\cdot\vec{n} = \frac{1}{\varepsilon_0}\sigma}$$

$$\nabla\times\vec{E} = \nabla\times\vec{E}_+\chi_+ + \nabla\times\vec{E}_-\chi_- + \nabla\chi_+\times\vec{E}_+ + \nabla\chi_-\times\vec{E}_- =$$

$$= \nabla\times\vec{E}_+\chi_+ + \nabla\times\vec{E}_-\chi_- + \delta_S\,\vec{n}_+\times\vec{E}_+ + \delta_S\,\vec{n}_-\times\vec{E}_- = 0$$

$$\Rightarrow \boxed{\vec{n}\times[\vec{E}] = 0}$$

Lineárne zdroje (1D)

$\to$ koncentrované na úsečke

$$\rho(x) = \lambda(x)\,\delta_\gamma \qquad Q = \int_\gamma \lambda\, dl \qquad \to \vec{E} \sim \frac{1}{2\pi\varepsilon_0}\frac{1}{R}\vec{e}_R \rightsquigarrow \text{singulárne na úsečke}$$

Bodové zdroje (0D)

$$\rho(x) = Q\,\delta(x-\vec{x}') \to \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{x}-\vec{x}'|^2}\vec{e}_r \quad \to \text{singulárne v } \vec{x}'$$

Vodiče

• obsahuje voľné náboje, kt. sa premiestnia na povrch vplyvom $\vec{E}$

$$\vec{E}_{povrch} + \vec{E}_{out} = 0 \qquad \vec{E}_{in} = 0 \Rightarrow \phi_{in} = konst$$

$$\int \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int \sigma \, dS \Rightarrow \vec{E} \cdot \vec{n} = \frac{\sigma}{\varepsilon_0}$$

• uzemnený vodič – spojený s oblasťi nulového potenciálu $\Rightarrow \phi_{vodič} = 0$

→ zmení tvar $\vec{E}$, aby $\vec{E} \parallel \vec{n}$ na vodiči
↳ ak dáme vodič na ekvipotenciálu ⇒ automaticky $\vec{E} \parallel \vec{n}$

Sila na vodič

• náboj rozložený na ploche, nespojitosť $\vec{E}$

← indukované pole

$$\vec{E} = \vec{E}_{out} + \vec{E}_{ind}$$

$$\vec{E}_{ind} = -\chi_+ \vec{E}^+_{ind} + \chi_- \vec{E}^-_{ind}$$
$$\vec{E}_{out} = \chi_+ \vec{E}^+_{out} + \chi_- \vec{E}^-_{out}$$

$$\Rightarrow \vec{E} = 0 + \chi_-(\vec{E}^-_{ind} + \chi \vec{E}^-_{out})$$

$$\Rightarrow \vec{E}_{ind} = \frac{1}{2}(-\chi_+ \vec{E} + \chi_- \vec{E})$$
$$\vec{E}_{out} = \frac{1}{2}\vec{E}$$

→ sila je len od vonkajšieho pola

$$\Delta \vec{f} = \Delta q \, \vec{E}_{out} = \Delta S \sigma \frac{1}{2} \vec{E} = \Delta S \frac{\varepsilon_0 E^2}{2} \vec{n}$$

⇒ sila na jednotku plochy $\boxed{\frac{\Delta \vec{f}}{\Delta S} = \vec{n} \frac{\varepsilon_0 E^2}{2}}$

Greenova funkce

$$\nabla^2 G(\vec{x}|\vec{x}') = \delta(\vec{x}|\vec{x}')$$ → fundamentálne riešenie

→ v $\mathbb{E}^3$ má tvar $G(\vec{x}|\vec{x}') = \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|}$ → platí iba pre celý priestor

→ riešenie Poissona $\nabla^2 \phi = -\frac{\rho(x)}{\varepsilon_0}$ bude potom

$$\phi = \int G(\vec{x}|\vec{x}') \frac{\rho(x')}{\varepsilon_0} d^3x' = \int \frac{1}{4\pi\varepsilon_0} \frac{\rho(x')}{|\vec{x} - \vec{x}'|} d^3x' = \frac{G * \rho(x)}{\varepsilon_0}$$

Dk:

$$\nabla^2\phi = \int \nabla^2 G(\vec{x}|\vec{x}') \, \frac{\rho(\vec{x}')}{\varepsilon_0} d^3x' = \int -\delta(\vec{x}|\vec{x}') \frac{\rho(\vec{x}')}{\varepsilon_0} d^3x' = -\frac{\rho(\vec{x})}{\varepsilon_0}$$

V oblasti

- riešime úlohu

$$\nabla^2 G_V(x|x') = \delta(x|x') \quad \text{v oblasti } V \subseteq \mathbb{E}^3$$
$$G_V(x|x') = 0 \quad \text{pro } x \in \partial V \;\; x' \in V$$

$$\Rightarrow \quad \phi(x) = \int_V G_V(x|x') \frac{1}{\varepsilon_0} \rho(x') dV$$

→ G_V vieme vyjadriť pomocou G v $\mathbb{E}^3$:

$$G_V = \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{1}{r(x|x')}}_{\text{G. fce pro } \mathbb{E}^3} + H(x|x')$$

↳ homogenne riešenie $\nabla^2 H = 0$, také aby splňovalo G_V podm. na ∂V

H je pole vnějších obrazů zrcadlících se přes hranici ∂V

Př: $$G_V(x|x') = \frac{1}{4\pi}\Big[\underbrace{\frac{1}{|\vec{x}-\vec{x}'|}}_{G \text{ v } \mathbb{E}^3} - \frac{a}{|\vec{x}'|} \underbrace{\frac{1}{|\vec{x}-\vec{x}''|}}_{H}\Big]$$ → kulová inverze

Metoda fiktivních nábojů

- riešime úlohu

(1) $\nabla^2\phi = -\frac{1}{\varepsilon_0} \rho_{in}(x)$ v oblasti V

(2) $\phi|_{\partial V} = \Phi$ na hranici

→ pokiaľ nájdeme ρ, t.ž. $\rho|_V = \rho_{in}$ tak budeme mať splnené (1)

$$\phi_{in} = \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{r(x|x')} \rho_{in}(x') dV'$$, ale $\phi_{in}|_{\partial V} \neq \Phi$

→ obecne ale nemusí G v $\mathbb{E}^3$ splňovať (2)

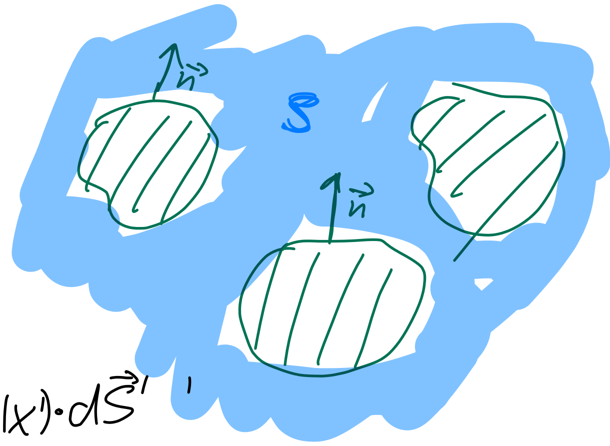
⇒ pridáme fiktivne náboje:

$$\phi_{out}(x) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{1}{r(x|x')} \rho_{out}(x') dV'$$

→ potom bude riešenie $\phi = \phi_{in} + \phi_{out}$

- pre prípad $\phi = 0$ dostaneme "zrkadlové" obrazy

Poissonova úloha s hodnotami na hranici



$$-\nabla^2\phi = \frac{1}{\varepsilon_0}\rho$$
$$\phi|_{\partial V} = \Phi$$

→ riešenie:

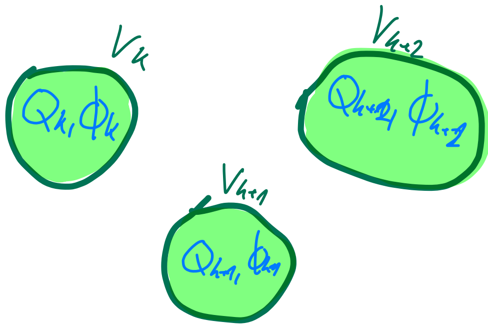
$$\phi(x) = \int_V G_V(x|x')\frac{1}{\varepsilon_0}\rho(x')\,dV' + \int_{\partial V}\Phi(x')\nabla G_V(x|x')\cdot d\vec{S}'$$

Dk: $\int G_V(x|x')\underbrace{\nabla^2\phi(x')}_{-\frac{1}{\varepsilon_0}\rho(x')}dV - \underbrace{\nabla^2 G_V(x|x')}_{\delta(x|x')}\phi(x')\,dV \overset{(GV)}{=} \int_{\partial V}\underbrace{G_V}_{0 \text{ na hranici}}\nabla\phi\cdot d\vec{S} - \underbrace{\phi}_{\Phi \text{ na hranici}}\nabla G_V\cdot d\vec{S}$

↳ pre $\rho = 0$ dostaneme Laplaceovu úlohu:

$$\nabla^2\phi = 0$$
$$\phi|_{\partial V} = \Phi \quad \Rightarrow \quad \phi(x) = \int \vec{n}\cdot\nabla G_V(x|x')\frac{1}{\varepsilon_0}\rho(x')\,dV'$$

Kapacity



- pole ϕ budené vodičmi
- pre vodič k platí

$$\Phi_k = \phi|_{\partial V_k} \quad \rightarrow \text{potenciál na vodiči}$$
$$Q_k = \int \sigma_k\,dS \quad \rightarrow \text{celkový náboj}$$

hustota náboja $\quad \sigma_k = -\varepsilon_0\vec{n}\cdot\nabla\phi|_{\partial V_k}$

- $\phi(\infty) = 0$
- medzi vodičmi $\nabla^2\phi = 0$

→ obecný potenciál môžeme napísať ako

$$\phi(\vec{x}) = \sum_{a=1}^{k} \underbrace{U_a}_{\leftarrow \text{potenciál na vodiči}} \Psi_a(\vec{x})$$

Ψ_a je homogénne riešenie $\nabla^2\Psi_a = 0$ splňujúce $\Psi_a(\partial V_b) = \delta_{ab}$

$$\Rightarrow Q_a = -\oint_{\partial V_a}\varepsilon_0\nabla\phi(\vec{x})\cdot d\vec{S} = \sum_{a=1}^{k} U_a\,\varepsilon_0\oint_{\partial V_a} -\nabla\Psi_b\cdot d\vec{S} = \sum_i U_a C_{ab}$$

$$C_{ab} = \oint_{\partial V_a} -\varepsilon_0\nabla\Psi_b\cdot d\vec{S}$$

• Symetria:

$$-\frac{1}{\varepsilon_0}(C_{ab}-C_{ba}) = \oint_{\partial V_a} \nabla\Psi_b \cdot d\vec{S} - \oint_{\partial V_b} \nabla\Psi_a \cdot d\vec{S} =$$

$$= \oint_{\partial\Omega} (\Psi_a \nabla\Psi_b - \Psi_b \nabla\Psi_a) \cdot d\vec{S} = \int_\Omega \Psi_a \overbrace{\nabla^2\Psi_b}^{0} - \Psi_b \overbrace{\nabla^2\Psi_a}^{0} \, dV$$

$$\Rightarrow C_{ab} = C_{ba}$$

• znamienka

↳ riešenie $\nabla^2\phi = 0$ má extrém na hranici (vodiči)

$\Rightarrow \nabla\Psi_a < 0$ na $\partial V_a \Rightarrow C_{aa} > 0$

$\Rightarrow \nabla\Psi_a > 0$ na ∂V_b $a \neq b \Rightarrow C_{ab} < 0$

Pomocou Greenovej fce

$$\left.\begin{matrix} \nabla^2\phi = 0 \\ \phi|_{\partial V_a} = \Phi_a \end{matrix}\right\} \quad \phi = \sum_b{}' \int_{\partial V_b} \underbrace{\Phi(x')}_{\text{konštanta na povrchu}} \nabla' G_V(x|x') \cdot d\vec{S} =$$

$$= \sum_b{}' \Phi_b \oint_{\partial V_b} \nabla' G_V(x|x') \cdot d\vec{S}$$

$$Q_a = \oint_{\partial V_a} -\frac{1}{\varepsilon_0} \nabla\phi \cdot d\vec{S} = \sum_b{}' \Phi_b \underbrace{\oint_{\partial V_a}\oint_{\partial V_b} -\frac{1}{\varepsilon_0} (\vec{n}_a\cdot\nabla)(\vec{n}_b\cdot\nabla') G_V(x|x') \, dS \, dS}_{C_{ab}}$$

Energia elektrostatického poľa

• idea – do rozmiestnených nábojov prinášame z ∞ náboj a meriame prácu

↳ 2 náboje:

$$W = \int \vec{F}\cdot d\vec{l} = \int_{-\infty}^{\vec{r}_2} q_2 \vec{E}_1 \cdot d\vec{l} = \int_{-\infty}^{\vec{r}_2} q_2 \, (-\nabla\phi) \cdot d\vec{l} = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} \frac{1}{4\pi\varepsilon_0}$$

↳ pre viacej nábojov: $W = \frac{1}{4\pi\varepsilon_0} \sum_a{}' \sum_{b<a} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|} = \frac{1}{4\pi\varepsilon_0} \frac{1}{2} \sum_a \sum_{a\neq b} \frac{q_a q_b}{|\vec{r}_a - \vec{r}_b|} =$

$$W = \frac{1}{2} \sum_a{}' q \, \phi(\vec{x}_a)$$

⤳ zo spojitíme:

$$W = \frac{1}{2} \int \rho(x) \phi(x) \, dV = \frac{1}{2} \int (\nabla\cdot\vec{E}) \varepsilon_0 \phi(x) \, dV =$$

$$= -\frac{1}{2} \varepsilon_0 \int \vec{E} \cdot \nabla\phi \, dV + \cancel{\frac{1}{2} \varepsilon_0 [\vec{E}\phi]_{-\infty}^{\infty}}^{0} = \frac{1}{2} \varepsilon_0 \int E^2 dV$$

→ pre vodiče: $Q_a = \sum C_{ab} U_b$

$$W = \sum_a{}' \frac{1}{2} \int \sigma_a \phi_a(x) \, dS_a = \frac{1}{2} \sum_a{}' C_{ab} U_a U_b$$

Jednoznačnosť riešenia Laplaca

Majme dve riešenia ϕ_1, ϕ_2 splňujúce $\phi_1|_{\partial\Omega} = \phi_2|_{\partial\Omega}$ potom:

$$0 = \int_\Omega (\phi_1 - \phi_2)\nabla^2(\phi_1 - \phi_2)\,dV = \oint_{\partial\Omega} \cancelto{0}{(\phi_1 - \phi_2)}\,\nabla(\phi_1 - \phi_2)\,d\vec{S} - \int_\Omega [\nabla(\phi_1 - \phi_2)]^2\,dV$$

$$\Rightarrow \phi_1 \overset{c}{=} \phi_2$$

Multipólový rozvoj

$$\partial_{i_1 \dots i_n} \frac{1}{|\vec{r} - \vec{r}'|}\Big|_{\vec{r}'=0} = \frac{M_{i_1 \dots i_n}(\vec{r})}{r^{2n+1}}$$

↳ rozvinieme $\frac{1}{|\vec{r} - \vec{r}'|}$ do Taylora:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \frac{1}{r} + \partial_i \frac{1}{|\vec{r} - \vec{r}'|}\Big|_{\vec{r}'=0} r_i' + \frac{1}{2!}\partial_{ij} \frac{1}{|\vec{r} - \vec{r}'|}\Big|_{\vec{r}'=0} r_i' r_j' + \dots =$$

$$= \frac{1}{r} + \frac{M_i(\vec{r})\, r_i'}{r^3} + \frac{1}{2!}\frac{M_{ij}(\vec{r})\, r_i' r_j'}{r^5} + \frac{1}{3!}\frac{M_{ijk}(\vec{r})\, r_i' r_j' r_k'}{r^7} + \dots$$

$M_i(\vec{r}) = r_i$
$M_{ij}(\vec{r}) = 3 r_i r_j - \delta_{ij} r^2$
$M_{ijk}(\vec{r}) = 15 r_i r_j r_k - 9\, \delta_{(ij} r_{k)}\, r^2$

- súčiny $r_i' r_j' \dots$ nemajú stopu, tak ich môžeme nahradiť $\frac{M_{ij\dots}(\vec{r})}{(2l+1)!!}$
- zároveň $M_{ij\dots}(\vec{r})$ môžeme nahradiť súčinom $r_{ij\dots} \cdot (2l+1)!!$

↓

$$4\pi\varepsilon_0 \phi = \frac{1}{r}\int \rho(x')\,dV + \frac{M_i}{r^3}\int r_i' \rho(x')\,dV + \frac{M_{ij}}{2!\,r^5}\int r_i' r_j' \rho(x')\,dV + \dots$$

$$= \frac{1}{r}\underbrace{\int \rho(x')\,dV}_{Q} + \underbrace{\frac{r_i}{r^3}}_{\frac{e_i}{r^2}} \underbrace{\int M_i(x')\rho(x')\,dV}_{Q_i} + \underbrace{\frac{r_i r_j}{2!\,r^5}}_{\frac{e_i e_j}{2!\,r^3}} \underbrace{\int M_{ij}(x')\rho(x')\,dV}_{Q_{ij}} + \dots$$

$$\Rightarrow \boxed{4\pi\varepsilon_0 \phi = \sum_{n=1}^{\infty} \frac{1}{n!}\frac{1}{r^n} Q_{i_1 \dots i_n} e_{i_1} e_{i_2} \dots e_{i_n}}$$

Sférický multipól

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2\vec{e}\cdot\vec{e}'rr'}} = \frac{1}{r}\frac{1}{\sqrt{\left(\frac{r'}{r}\right)^2-2\vec{e}\cdot\vec{e}'\frac{r'}{r}+1}}$$

$$= \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\vec{e}\cdot\vec{e}') = \sum_{\ell=0}^{\infty}\frac{r'^{\ell}}{r^{\ell+1}} P_\ell(\cos\vartheta)$$

$$P_\ell(\vec{e}\cdot\vec{e}') = \sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1} Y_\ell^m(\vec{e})\, Y_\ell^{m*}(\vec{e}')$$

$$\Rightarrow \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r'^{\ell}}{r^{\ell+1}}\frac{4\pi}{2\ell+1} Y_\ell^m(\vec{e})\, Y_\ell^{m*}(\vec{e}')$$

$$\Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sqrt{\frac{4\pi}{2\ell+1}}\,\frac{Y_\ell^m(\vec{e})}{r^{\ell+1}}\underbrace{\int\sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^{m*}(\vec{e}')\, r'^{\ell}\rho(x')\,dV}_{M_\ell^m}$$

M_ℓ^m → sférický multipól

Separace Laplaca v sféríckách

$$\nabla^2 = \frac{1}{r^2\sin\vartheta}\left[\frac{\partial}{\partial r}\left(r^2\sin\vartheta\frac{\partial\cdot}{\partial r}\right) + \frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\cdot}{\partial\vartheta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\vartheta}\frac{\partial\cdot}{\partial\varphi}\right)\right] =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

$$\phi = R(r)\,\Theta(\vartheta)\,\Psi(\varphi)$$

$$\frac{\nabla^2\phi}{\phi} = \underbrace{\frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\frac{1}{R}}_{=\text{konst}} + \frac{1}{\cancel{r^2}\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right)\frac{1}{\Theta} + \frac{1}{\cancel{r^2}\sin^2\vartheta}\frac{\partial^2\Psi}{\partial\varphi^2}\frac{1}{\Psi} = 0$$

$$(r^2R')'\frac{1}{R} = \lambda \rightarrow (2rR' + r^2R'') = \lambda R \rightarrow R'' + \frac{2}{r}R' + \frac{\lambda R}{r^2} = 0$$

$$R = r^\ell:\quad \ell(\ell-1)\cancel{r^{\ell-2}} + 2\ell\cancel{r^{\ell-2}} + \lambda\cancel{r^{\ell-2}} = 0$$

$$\ell^2 - \ell + 2\ell = \lambda \Rightarrow \lambda = \ell(\ell+1)$$ → druhé riešenie pre $-(\ell+1)$

$\Rightarrow$ riešenie má tvar $R = Ar^\ell + \frac{B}{r^{\ell+1}}$

$$(\ell+1)\ell\sin^2\vartheta + \sin\vartheta\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right)\frac{1}{\Theta} + \underbrace{\frac{\partial^2\Psi}{\partial\varphi^2}\frac{1}{\Psi}}_{\text{konst}} = 0$$

$$\Rightarrow \frac{\Psi''}{\Psi} = -m^2 \Rightarrow \Psi'' + m^2\Psi = 0 \Rightarrow \Psi = A_m e^{im\varphi}$$

$$\Psi(0) = \Psi(2\pi) \Rightarrow m \in \mathbb{Z}$$

$$\sin\vartheta(\sin\vartheta\,\Theta')' + (\ell+1)\ell\sin^2\vartheta - m^2 = 0$$

↳ riešenia majú tvar $\Theta(\vartheta) = P_m^\ell(\cos\vartheta)$; $P_\ell(x) = \frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

$$\Rightarrow \phi_{m\ell} = \left(\frac{A}{r^{\ell+1}} + Br^\ell\right)P_m^\ell(\cos\vartheta)e^{im\varphi} \qquad |m| \le \ell$$

${}_1\phi_{m\ell} = r^\ell Y_m^\ell(\vartheta)$ ⤳ mód pre $r \sim 0$ $\quad Y_m^\ell = C_{m\ell}P_m^\ell(\cos\vartheta)e^{im\varphi}$

↳ ON bázové fce na sfére

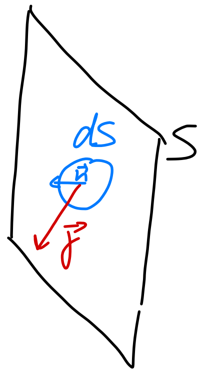
${}_2\phi_{m\ell} = \frac{Y_m^\ell(\vartheta)}{r^{\ell+1}}$ ⤳ mód pre $r \sim \infty$ → tj. multipól

↳ obecné riešenie pre $r \sim \infty$ bude $\phi = \sum_{\ell,|m|\le\ell} U_{m\ell}\frac{Y_m^\ell(\vartheta)}{r^{\ell+1}}$

↳ pre azimutálne symetrické: $\phi = \sum_\ell \frac{P_\ell(\cos\vartheta)}{r^{\ell+1}}U_\ell'$

Magnetostatika

Tok náboja



$$I = \int_S \vec{j}\cdot d\vec{S}$$ → celkový prúd

$$\frac{dQ_{in}}{dt} = -I_{out} \iff \frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

zákon zachovania náboja ⟺ rovnica kontinuity toku náboja

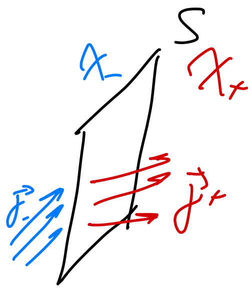
- systém častíc s rýchlosťou $\vec{v}$ a nábojom ρ: $\vec{j} = \rho\vec{v}$ konvekčný prúd

→ stacionárna situácia $\frac{\partial\rho}{\partial t} = 0 \Rightarrow \nabla\cdot\vec{j} = 0$

Singulárne zdroje

- regulárni zdroje (3D) → $\vec{j}$ je spojitá
- plošné toky → $\vec{j} = \vec{i}\,\delta_S$ → $I = \int \vec{i}\cdot d\vec{l}$
- tenké vodiče → $\vec{j} = I\,\delta_\gamma\,\vec{e}_\gamma$ ← tečna ku γ

Elektrické pole vodiče



$\gamma\vec{E} = \vec{j}$ → Ohmický vodič

↳ na rozhraní:

$\nabla\cdot\vec{j} = 0 \Rightarrow \vec{n}\cdot[\vec{j}] = 0 \Rightarrow \gamma_+\vec{n}\cdot\vec{E}_+ = \gamma_-\vec{n}\cdot\vec{E}_-$ → nespojitosť vodivosti

(→ sp. prúdu)

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{n}\cdot[\vec{E}] = \frac{\sigma}{\varepsilon_0}$

→ pre rozhranie vákuum vodič:

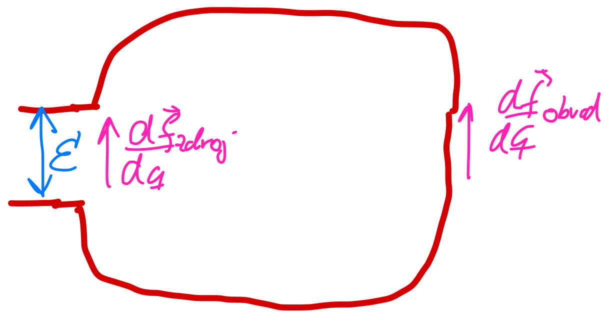
$\vec{n}\cdot\vec{j}_{vod} = 0 \qquad \vec{n}\cdot\vec{E}_{vod} = 0$

Elektromotorická sila

- pridávaná zdrojom na udržanie prúdu

$$\mathcal{E} = \int_\gamma \frac{d\vec{f}_{zdroj}}{dq} \cdot d\vec{s}$$

$$\vec{E}_{bat} + \frac{d\vec{f}_{zdroj}}{dq} = 0 \Rightarrow \vec{E}_{bat} = -\frac{d\vec{f}_{zdroj}}{dq}$$



Formulace magnetostatiky

$$\left.\begin{aligned} &\nabla\cdot\vec{B} = 0 \\ &\nabla\times\vec{B} = \mu_0 \vec{j} \\ &\vec{f} = \rho\vec{v}\times\vec{B} = \vec{j}\times\vec{B} \end{aligned}\right\} \text{rovnice magnetostatiky}$$

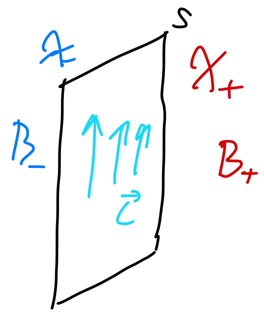
Amperov zákon

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I}$$

Magnetický tok

$$\boxed{\Psi = \int_S \vec{B}\cdot d\vec{S}}$$

Plošné zdroje



$$\vec{j} = \vec{\iota}\,\delta_S$$

$$\vec{B} = \vec{B}_+\chi_+ + \vec{B}_-\chi_-$$

$$\nabla\cdot\vec{B} = (\dots) + \vec{n}_+\cdot\vec{B}_+\delta_S + \vec{n}_-\cdot\vec{B}_-\delta_S = 0 \Rightarrow \vec{n}\cdot[\vec{B}] = 0$$

$$\nabla\times\vec{B} = (\dots) + \vec{n}_+\times\vec{B}_+\delta_S + \vec{n}_-\times\vec{B}_-\delta_S = \vec{\iota}\,\delta_S\mu_0 \Rightarrow \vec{n}\times[\vec{B}] = \vec{\iota}\mu_0$$

Potenciál

$$\nabla\cdot\vec{B} = 0 \Leftrightarrow \vec{B} = \nabla\times\vec{A} \xrightarrow{\text{nejednoznačné}} \vec{A}' = \vec{A} + \nabla f$$

$$\Psi = \int_S \vec{B}\cdot d\vec{S} = \int_S \nabla\times\vec{A}\cdot d\vec{S} = \oint_{\partial S} \vec{A}\cdot d\vec{l}$$

{ využime k splneniu $\nabla\cdot\vec{A} = 0$ (Coulomb)

$$\nabla\times\vec{B} = \nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{j} \Rightarrow \boxed{-\nabla^2\vec{A} = \mu_0\vec{j}}$$

↳ 3 rovnice pre 3 zložky

- v prípade kartézskych súradníc:

$$\vec{A} = \mu_0 G * \vec{j}, \text{ kde } G = \frac{1}{4\pi}\frac{1}{r} \Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}dV}$$

$$\nabla\cdot\vec{A} = \mu_0\nabla\cdot(G*\vec{j}) = \mu_0\nabla\cdot(\vec{j}*G) = \mu_0(\nabla\cdot\vec{j})*G = 0$$

Biot-Savart

$$\vec B = \nabla\times\vec A = \mu_0 \nabla\times(G*\vec j) = \frac{\mu_0}{4\pi}\int \nabla\frac{1}{|\vec r-\vec r'|}\times \vec j'\, dV' = \frac{\mu_0}{4\pi}\int -\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}\times\vec j\, dV =$$

$$\vec B = \frac{\mu_0}{4\pi}\int \vec j\times\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}\, dV$$

→ pre tenký vodič: $\vec j = I\,\delta_\gamma\, \vec e_\gamma$: $\vec B = \frac{\mu_0 I}{4\pi}\int \vec e_\gamma\times\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}\, dl =$

$$\vec B = \frac{\mu_0 I}{4\pi}\int_\gamma d\vec l\times \frac{\vec r-\vec r'}{|\vec r-\vec r'|^3}$$

Multipólový rozvoj

$$\vec A = \frac{\mu_0}{4\pi}\int \underbrace{\frac{1}{|\vec r-\vec r'|}}_{\frac1r+\frac{r_i}{r^3}r_i'+\dots}\vec j(\vec r)\,dV = \frac{\mu_0}{4\pi}\frac1r\underbrace{\int\vec j\, dV}_{\text{monopól}} + \frac{\mu_0}{4\pi}\frac{r_i}{r^3}\underbrace{\int r_i'\,\vec j(x)\,dV}_{\text{dipól } I_{ij}}+\dots$$

$$\int\vec j\, dV = \int \vec j\cdot\nabla\vec r + \nabla\cdot\vec j\,\vec r\, dV = \int\nabla\cdot(\vec j\,\vec r)\, dV = \int_{\partial V}\vec j\,\vec r\, dV = 0$$

$\vec j$ 0 na ∂V

$$I_{ij}+I_{ji} = \int r_i j_j + r_j j_i\, dV = \int r_i j_k\,\partial_k r_j + r_j j_k\,\partial_k r_i + \partial_k j_k\, r_i r_j\, dV =$$

$$= \int\partial_k(r_i r_j j_k)\, dV = \int_{\partial V} r_i r_j \underbrace{j_k}_{0}\, n_k\, dS = 0$$

$$\Rightarrow I_{ij} = -I_{ji} \quad\Rightarrow\quad I_{ij} = \varepsilon_{ijk}\,m^k \quad\Rightarrow\quad m^k = -\frac12\varepsilon_{kij}\,I_{ij}$$

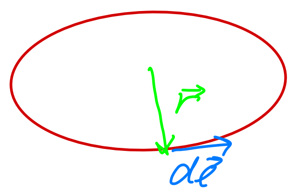
$$\Rightarrow A^j = \frac{\mu_0}{4\pi}\frac{r_i\,\varepsilon_{ijk}\,m^k}{r^3}+\dots = \frac{\mu_0}{4\pi}\varepsilon_{jki}\frac{m^k r_i}{r^3}+\dots \Rightarrow \vec A = \frac{\mu_0}{4\pi}\frac{\vec m\times\vec r}{r^3}$$

Dipól

$$m^k = \frac12\varepsilon_{kij}\,I_{ij} = \frac12\varepsilon_{kij}\int r_i\, j_j\, dV \Rightarrow \vec m = \frac12\int\vec r\times\vec j\, dV$$

• pre $\vec j = I\,\delta_\gamma\,\vec e_\gamma$ ⤳ uzavretá smyčka

$$\vec m = \frac I2\oint_\gamma \vec r\times d\vec l = I\int d\vec S = IS\,\vec n$$



Sila na dipól:

$$\vec F = \int\vec j\times\vec B\, dV = \int\vec j\times\left(\vec B|_{x_0} + \vec r\cdot\nabla\vec B|_{x_0}\right)dV =$$

$$= \int \cancelto{0}{\vec{j}\, dV} \times \vec{B}\Big|_{x_0} + \int \vec{j} \times \left(\vec{r} \cdot \nabla \vec{B}\Big|_{x_0}\right) dV$$

$$F_i = \int \varepsilon_{ijk}\, j_j\, x_m \partial_m B_k \Big|_{x_0} dV = \varepsilon_{ijk} \underbrace{\int j_j x_m\, dV}_{\varepsilon_{jmn} m^n} \partial_m B_k \big|_{x_0} = \varepsilon_{ijk}\, \varepsilon_{jmn}\, m^n\, \partial_m B_k =$$

$$= \partial_i (m^k B_k) + \cancelto{0}{m_i \partial_k B_k} \Rightarrow \vec{F} = \nabla(\vec{m} \cdot \vec{B}) = \vec{m} \cdot \nabla \vec{B}$$

$$\vec{M} = \int \vec{r} \times (\vec{j} \times \vec{B}) dV = \int \vec{r} \times (\vec{j} \times \vec{B}|_{x_0})\, dV = \int \vec{j}\,(\vec{r} \cdot \vec{B}|_{x_0}) dV - \cancelto{0}{\int \vec{B}|_{x_0} \underbrace{(\vec{r} \cdot \vec{j})}_{\text{antisym.}} dV} =$$

$$= \int \vec{j}\, \vec{r}\, dV \cdot \vec{B}|_{x_0} = \vec{m} \times \vec{B}$$

Systém smyček

$$\vec{B} = \sum_k \frac{\mu_0 I_k}{4\pi} \int_{\gamma_k} \frac{d\vec{s}\,' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \qquad \vec{j} = \sum_k \vec{e}_{\gamma_k} \delta_{\gamma_k} I_k$$

$$\vec{A} = \sum_k \frac{\mu_0 I_k}{4\pi} \int_{\gamma_k} \frac{d\vec{\ell}\,'}{|\vec{r} - \vec{r}\,'|}$$

$$\Psi_{\text{skrze } \gamma_1 \text{ od } \gamma_2} = \int_{S_1} \vec{B}_2 \cdot d\vec{S} = \int_{\gamma_1} \vec{A}_2 \cdot d\vec{s}_1 = I_2 \underbrace{\frac{\mu_0}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \frac{d\vec{s}_2 \cdot d\vec{s}_1}{|\vec{r}_2 - \vec{r}_1|}}_{L_{12}}$$

$$\Rightarrow L_{\ell k} = \frac{\mu_0}{4\pi} \int_{\gamma_\ell} \int_{\gamma_k} \frac{d\vec{s}_k \cdot d\vec{s}_\ell}{|\vec{x}_\ell - \vec{x}_k|} \qquad \Psi_\ell = L_{\ell k} I_k$$

↳ samoindukčnosť: $L_{kk} = \frac{\mu_0}{4\pi I_k^2} \int_{\gamma_1} \int_{\gamma_2} \frac{d\vec{s}_1 \cdot d\vec{s}_2}{|\vec{r}_1 - \vec{r}_2|}$ → diverguje pre nekonečne tenký vodič, treba počítať konečnú šírku

Sila medzi smyčkami

$$\vec{F}_{12} = \int \vec{j}_1 \times \vec{B}_2\, dV = I_1 \int_{\gamma_1} d\vec{s}_1 \times \vec{B}_2 = \frac{I_1 I_2 \mu_0}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \frac{\overset{a}{d\vec{s}_1} \times \overset{b}{d\vec{s}_2} \times \overset{c}{(\vec{r}_1 - \vec{r}_2)}}{|\vec{r}_1 - \vec{r}_2|^3} =$$

$$= \frac{I_1 I_2 \mu_0}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \underbrace{\frac{d\vec{s}_2 \left(d\vec{s}_1 \cdot (\vec{r}_1 - \vec{r}_2)\right)}{|\vec{r}_1 - \vec{r}_2|^3}}_{\int_{\gamma_1} d\vec{s}_1 \cdot \nabla \frac{1}{|\vec{r}_1 - \vec{r}_2|} = \int_{S_1} \nabla \times \nabla (..)\, dS_1 = 0} - \frac{(\vec{r}_1 - \vec{r}_2)(d\vec{s}_1 \cdot d\vec{s}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$= \frac{I_1 I_2 \mu_0}{4\pi} \int_{\gamma_1} \int_{\gamma_2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_1 - \vec{r}_2|^3}\, d\vec{s}_1 \cdot d\vec{s}_2$$

Kvazistacionárne priblíženie

$$\nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{B}=\mu_0\vec{j}$$
$$\nabla\cdot\vec{E}=\frac{\varrho}{\varepsilon_0} \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

Faradayov zákon

Premenné magnetické pole

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \Big/ \int_S d\vec{S}$$

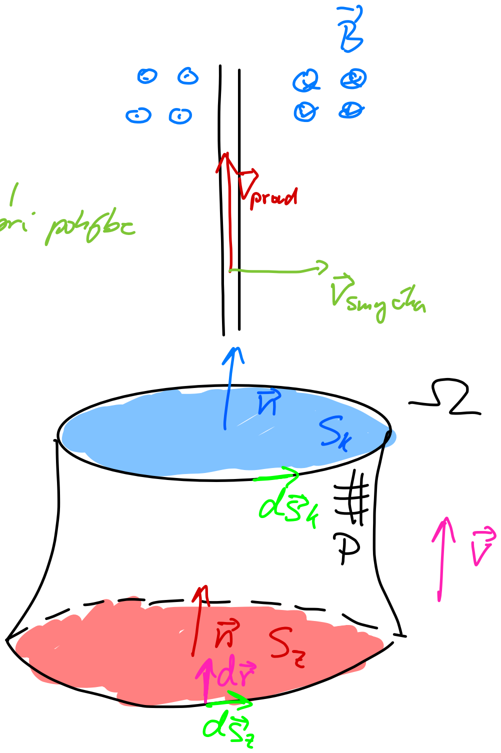
~> zintegrujeme cez obsah smyčky

$$\oint_\gamma \underbrace{\vec{E}}_{\frac{d\vec{f}}{dq}}\cdot d\vec{\ell}=-\frac{d}{dt}\int\vec{B}\cdot d\vec{S}=-\frac{d\Psi}{dt} \quad\Rightarrow\quad \mathcal{E}=-\frac{d\Psi}{dt}$$

Pohyb smyčky

$$0=\nabla\cdot\vec{B} \quad \Big/ \int_\Omega dV$$

← integrace cez objem, kt. vytvorí smyčka pri pohybe



$$=\int_{S_k}\vec{B}\cdot d\vec{S}-\int_{S_z}\vec{B}\cdot d\vec{S}+\int_P\vec{B}\cdot d\vec{S}$$

$$=\Psi_k-\Psi_z+\int_P\vec{B}\cdot(d\vec{s}\times\underbrace{d\vec{r}}_{\vec{v}_{smyčka}dt})=$$

$$=\Delta\Psi+\int_{t_z}^{t_k}\oint_\gamma d\vec{s}\cdot\underbrace{(\vec{v}_{smyčka}\times\vec{B})}_{\frac{d\vec{f}}{dq}}dt=\Delta\Psi+\int_{t_z}^{t_k}\mathcal{E}\,dt$$

$$\Rightarrow\quad \mathcal{E}=-\frac{d\Psi}{dt}$$

~> celkovo potom dostaneme $\displaystyle -\frac{d\Psi}{dt}=\oint_\gamma\left(\frac{d\vec{f}_E}{dq}+\frac{d\vec{f}_B}{dq}\right)\cdot d\vec{s}=\oint_\gamma(\vec{E}+\vec{v}\times\vec{B})\cdot d\vec{s}$

Práce vykonaná proudem ve smyčce

→ práce, kt. vykoná zdroj pri posune smyčky, zmene polí.

$$\frac{dA_{zdroj}}{dt}=\underbrace{-}_{\text{pôsobí proti zmene}}\frac{dq}{dt}\mathcal{E}=I\frac{d\Psi}{dt} \quad \rightsquigarrow \quad \Delta A_{zdroj}=I\,\Delta\Psi$$

↳ práce se koná pri zmene Ψ

Energie jednej smyčky

$$\frac{dA_{zdroj}}{dt} = \frac{dU_s}{dt} = I_s \frac{d\Psi_s}{dt} = I_s \frac{d}{dt}\left(I_s L_{ss}\right) = \frac{1}{2} L_{ss} \frac{dI_s^2}{dt}$$

$$\rightsquigarrow \boxed{U_s = \frac{1}{2} L_{ss} I_s^2}$$

→ energia jednej smyčky je daná prácou potrebnou na "naštartovanie" prúdu

Energie systému smyček

→ indukciou ukážeme $\quad U = \frac{1}{2} \sum_{l,k} L_{lk} I_l I_k$

Dk (indukciou):

- pre jednu smyčku je $\quad U_1 = \frac{1}{2} L_{ss} I_s^2$

- indukčný krok

↳ do systému smyčiek pridáme ďalšiu:

$$\Psi_s = L_{ss} I_s + \sum_l L_{sl} I_l \quad \rightarrow \text{nová}$$

$$\Psi_k = L_{ks} I_k + \sum_l L_{kl} I_l \quad \rightarrow \text{staré}$$

↳ pri presune dôjde len ku zmene $\frac{dI_s}{dt}$, ostané sú nulové

$$\frac{dU_s}{dt} = I_s \frac{d\Psi_s}{dt} + \sum_k I_k \frac{d\Psi_k}{dt} = L_{ss} I_s \frac{dI_s}{dt} + \sum_k I_k L_{ks} \frac{dI_s}{dt}$$

$$\Rightarrow \Delta U_s = \frac{1}{2} I_s^2 L_{ss} + \sum_k I_k \underbrace{L_{ks}}_{\frac{1}{2}(L_{ks} + L_{sk})} I_s$$

$$U_{celk} = \frac{1}{2} I_s^2 L_{ss} + \frac{1}{2} \sum_k I_k L_{ks} I_s + \frac{1}{2} \sum_k I_s L_{sk} I_k + \frac{1}{2} \sum_{l,m} I_m L_{lm} I_l$$

$$= \frac{1}{2} \sum_m I_m L_{lm} I_l$$

$\square$

Lokální zákony zachování

Rovnice kontinuity

veličina na konci − veličina na začátku + tok veličiny = množství vznikající v tělese

$$\int w|_{end} \, dV - \int w|_{beg} \, dV + \int_{t_{beg}}^{t_{end}} \int_{\partial V} \vec{w} \cdot d\vec{S} \, dt = \int_{t_{beg}}^{t_{end}} \int s \, dV \, dt$$

(diff. tvar $\vec{w} = w\vec{v}$

hust. toku veličiny

$$\frac{\partial w}{\partial t} + \nabla \cdot \vec{w} = s \quad \rightsquigarrow \quad \nabla_\mu w^\mu = s$$

Zákon zachování náboje

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \quad / \nabla \cdot \qquad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$0 = \mu_0 \nabla \cdot \vec{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \cdot \vec{E} = \mu_0 \nabla \cdot \vec{j} + \frac{1}{\varepsilon_0 c^2} \frac{\partial}{\partial t} \rho \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\nabla_\mu j^\mu = 0$$

Bilance energie

výkon

$$W = \vec{j} \cdot \vec{E} = \varepsilon_0 c^2 \vec{E} \cdot \left(\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) = \varepsilon_0 c^2 \nabla \times \vec{B} \cdot \vec{E} - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} =$$

$$= \varepsilon_0 c^2 \nabla \cdot (\vec{B} \times \vec{E}) + (\nabla \times \vec{E}) \cdot \vec{B} \varepsilon_0 c^2 - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \overset{?}{=}$$

$$= -\varepsilon_0 c^2 \nabla \cdot (\vec{E} \times \vec{B}) - \varepsilon_0 c^2 \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \varepsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} =$$

$$= -\nabla \cdot \vec{S} - \frac{\partial}{\partial t} \left(\frac{1}{2\mu_0} B^2 + \frac{1}{2} \varepsilon_0 E^2 \right) =$$

$$= -\nabla \cdot \vec{S} - \frac{\partial u}{\partial t}$$

$$\vec{S} = \varepsilon_0 c^2 \vec{E} \times \vec{B} \qquad \Rightarrow \quad -W = \nabla \cdot \vec{S} + \frac{\partial u}{\partial t}$$

$$u = \frac{1}{2\mu_0} B^2 + \frac{1}{2} \varepsilon_0 E^2$$

Bilance hybnosti

$$\vec{f} = \rho \vec{E} + \vec{j} \times \vec{B} \Rightarrow$$

$$f_i = \rho E_i + \varepsilon_{ijk} j_j B_k = \varepsilon_0 E_i \partial_j E_j + \varepsilon_{ijk} B_k \left(\varepsilon_0 c^2 \varepsilon_{jab} \partial_a B_b - \varepsilon_0 \partial_t E_j \right) =$$

$$= \varepsilon_0 E_i \partial_j E_j + \varepsilon_0 c^2 (\delta_i^b \delta_k^a - \delta_i^a \delta_k^b) B_k \partial_a B_b - \varepsilon_0 \partial_t (\varepsilon_{ijk} B_k E_j) + \varepsilon_0 \underbrace{\varepsilon_{ijk}}\, E_j \underbrace{\partial_t B_k}_{-\varepsilon_{kab}\partial_a E_b} =$$

$$= \varepsilon_0 E_i \partial_j E_j + \varepsilon_0 c^2 B_k \partial_k B_i - \varepsilon_0 c^2 B_k \partial_i B_k - \varepsilon_0 E_j \partial_i E_j + \varepsilon_0 E_j \partial_j E_i \quad (\delta_i^a \delta_j^b - \delta_i^b \delta_j^a)$$
$$- \varepsilon_0 \partial_t (\varepsilon_{ijk} E_j B_k)$$

$$= \varepsilon_0 E_i \partial_j E_j + \varepsilon_0 E_j \partial_j E_i - \varepsilon_0 E_j \partial_i E_j$$
$$\underbrace{\varepsilon_0 c^2 B_i \partial_j B_j + \varepsilon_0 B_j \partial_j B_i}_{\varepsilon_0 \partial_j (B_i B_j c^2 + E_i E_j)} \underbrace{- \varepsilon_0 B_j \partial_i B_j c^2}_{\frac{1}{2}\partial_j (\varepsilon_0 c^2 B_k B_k + \varepsilon_0 E_k E_k)\delta_{ij}} - \varepsilon_0 \partial_t (\varepsilon_{ijk} E_j B_k)$$

$$\Rightarrow \vec{f} = -\varepsilon_0 \frac{\partial \vec{g}}{\partial t} - \nabla \cdot \overleftrightarrow{T} \qquad \overleftrightarrow{T} = -\overleftrightarrow{\sigma}$$
$$\overleftrightarrow{\sigma} = \varepsilon_0 \left(c^2 \vec{B} \otimes \vec{B} + \vec{E} \otimes \vec{E} - \frac{1}{2} \overleftrightarrow{\delta} (E^2 + c^2 B^2)\right)$$

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} = \left(F^{\mu\kappa} F^{\nu}{}_{\kappa} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \eta^{\mu\nu}\right) \varepsilon_0 c^2$$

Časovo premenné pole

Formulace elektrodynamiky

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Potenciály

$$\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \Rightarrow \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla \cdot \vec{E} = -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \underbrace{\nabla \times \nabla \times \vec{A}}_{\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \phi + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{j}$$

$$\Rightarrow -\nabla^2 \phi - \frac{\partial}{\partial t} \nabla \cdot \vec{A} = \frac{\rho}{\varepsilon_0} \qquad -\nabla^2 \vec{A} + \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}\right) + \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = \mu_0 \vec{j}$$

$$\Downarrow \qquad \Downarrow$$

$$-\Box \phi - \frac{\rho}{\varepsilon_0} + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}\right) \qquad -\Box \vec{A} = \mu_0 \vec{j} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}\right)$$

Kalibrace

$$\left.\begin{aligned} \vec{A}' &= \vec{A} + \nabla\psi \\ \phi' &= \phi - \frac{\partial\psi}{\partial t} \end{aligned}\right\}$$ riešenie pre $\vec{E}, \vec{B}$ sa nezmení

Lorenzova kalibrace

• chceme $\nabla\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial\phi'}{\partial t} = 0$

$$\Downarrow$$

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla^2\psi - \frac{1}{c^2}\frac{\partial\psi}{\partial t} = 0 \Rightarrow \Box\psi = -\nabla\cdot\vec{A} - \frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

Potom:

$$\left.\begin{aligned} \Box\phi &= -\frac{\rho}{\varepsilon_0} \\ \Box\vec{A} &= -\mu_0\vec{j} \end{aligned}\right\} \Box A^\mu = -\mu_0 j^\mu$$

→ ψ môžeme ešte dodatočne meniť: $\psi' = \psi + \chi$, kde $\Box\chi = 0$

Coulombova kalibrace

$\nabla\cdot\vec{A}' = 0 \quad \Rightarrow \quad \nabla^2\psi = -\nabla\cdot\vec{A}$

↳ potom:

$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$ ⇝ nekauzálne

$\Box\vec{A} = -\mu_0\vec{j} - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi$ → priamo ovplyvňuje $\vec{E}$ ⇒ odstránenie nekauzality v $\vec{E}$

Weylova kalibrace

$\phi = \Phi$ → libovolná funkce

⇒ $\phi = \phi' - \frac{\partial\psi}{\partial t}$ ⇒ $\psi = \int \phi' - \Phi \, dt$ → lze vždy najísť

→ volbou $\nabla^2\Phi = -\frac{\rho}{\varepsilon_0}$ dostaneme Coulomba

Riešenie nehomogénnej vlnovej rovnice

↳ hľadáme riešenie rovnice $-\Box A^\mu = \mu_0 j^\mu$ ⤳ Lorenzova kalibrace

↳ pomocou Greenovej fce $-\Box G(x|x') = \delta(x|x')$

Potom bude riešenie $A^\mu = \mu_0 \int G(x|x')\, j^\mu(x')\, d\Omega$

↳ určená až na homog. riešenie $\Box A^\mu = 0$ ⤳ EM vlny

Požiadavky na G:

- $G(x|x') = G(\Delta x)$ $\Delta x = x - x'$ ⤳ Poincarého (translační) invariance
- $G(x|x') = G(\Delta x^2)$ $\Delta x^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$ ⤳ Lorentzova (boosty a rotace) invariance
- $G_\mu^\nu = G \delta_\mu^\nu$ ⤳ obecný tenzorový charakter daný glob. rovnobežnosťou Minkowského

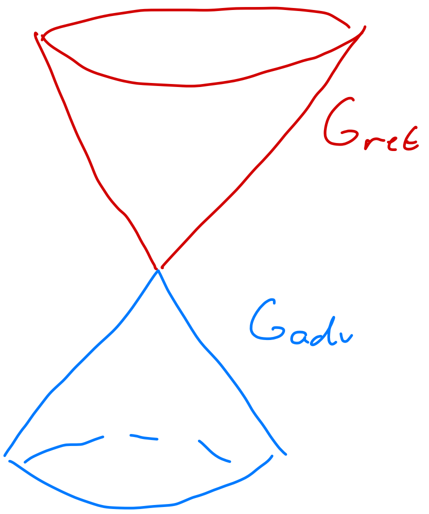
Riešenie $-\Box G = \delta$

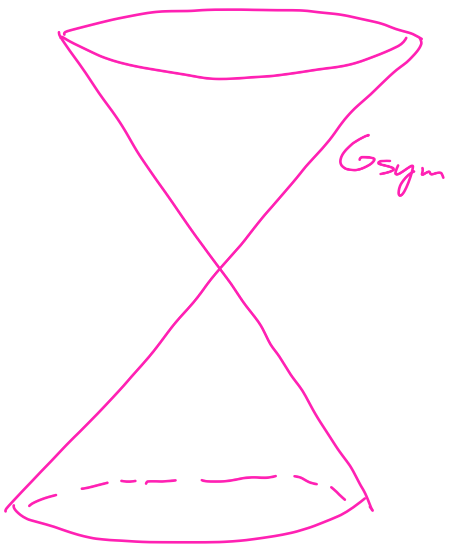
$$\delta(f(x)) = \sum_{x_0 : f(x_0) = 0} \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

↳ $$G_{sym} = \frac{1}{8\pi r}\left[\delta(r - c\Delta t) + \delta(r + c\Delta t)\right] = \frac{1}{4\pi} \delta(\Delta x^2)$$

$$G_{ret} = \frac{1}{4\pi r} \delta(r - c\Delta t) = \frac{1}{2\pi} \delta(\Delta x^2)\, \Theta(\Delta t)$$

$$G_{adv} = \frac{1}{4\pi r} \delta(r + c\Delta t) = \frac{1}{2\pi} \delta(\Delta x^2)\, \Theta(-\Delta t)$$





$$G_{sym} = \frac{1}{2}(G_{sym} + G_{adv})$$

$$G_c = G_{ret} - G_{adv}$$

↑ kauzální, teda rieší bez zdrojoví $\Box G_c = 0$ ⤳ vlny

Retardovaný potenciál

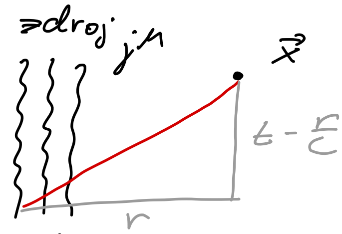
$$t' = t - \frac{|\vec r - \vec r'|}{c} = t - \frac{r}{c}$$

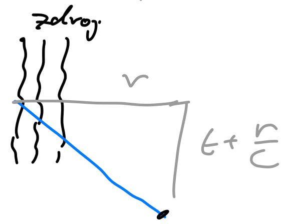
$$A^\mu_{ret} = \mu_0 \int G_{ret}\, j^\mu \, dV = \frac{\mu_0}{4\pi} \int \frac{j^\mu(\vec r', t')}{|\vec r - \vec r'|} \delta\big(c(t-t') - |\vec r - \vec r'|\big)\, dV =$$

$$= \frac{\mu_0}{4\pi} \int \frac{j^\mu\left(\vec r', t - \frac{|\vec x - \vec x'|}{c}\right)}{|\vec x - \vec x'|}\, d^3\vec x'$$

$$\Rightarrow \quad \vec A_{\substack{ret\\adv}} = \frac{\mu_0}{4\pi} \int \frac{\vec j\left(\vec x', t \mp \frac{|\vec x - \vec x'|}{c}\right)}{|\vec x - \vec x'|}\, d^3\vec x'$$

$$\phi_{\substack{ret\\adv}} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho\left(\vec x', t \mp \frac{|\vec x - \vec x'|}{c}\right)}{|\vec x - \vec x'|}\, d^3\vec x'$$





Jefimenkove vzťahy

$$\vec E = -\nabla\phi - \frac{\partial \vec A}{\partial t} = -\frac{1}{4\pi\varepsilon_0} \int \underbrace{\frac{\vec e_r}{r^2}\rho\Big|_{t_{ret}}}_{\text{Coulomb}} + \underbrace{\frac{1}{cr}\frac{\partial \rho}{\partial t}\Big|_{t_{ret}}\vec e_r + \frac{1}{c^2 r}\frac{\partial \vec j}{\partial t}\Big|_{t_{ret}}}_{\text{žiarenie}}\, dV$$

$$\vec B = \nabla \times \vec A = \frac{\mu_0}{4\pi} \int \underbrace{\vec j\Big|_{t_{ret}} \times \frac{\vec e_r}{r^2}}_{\text{Biot-Savart}} + \underbrace{\frac{\partial \vec j}{\partial t}\Big|_{t_{ret}} \times \frac{\vec e_r}{cr}}_{\text{žiarenie}}\, dV$$

Liénard-Wiechertove potenciály

↳ pole bodového náboja

$$j^\mu = \int u^\mu q\, \delta(x|x_i(\tau))\, c\, d\tau$$

$$A^\mu = \int G_{ret}(x|x')\, j^\mu(x')\, d^4\Omega = \frac{1}{2\pi\varepsilon_0 c^2} \iint \delta\big((x-x')^2\big)\, \Theta(\Delta t)\, u^\mu c q\, \delta(x|x_i(\tau))\, d\tau\, d^4\Omega =$$

$$= \frac{cq}{2\pi\varepsilon_0 c^2} \int \delta\big((x - x_i(\tau))^2\big)\, \Theta(\Delta t)\, u^\mu(\tau)\, d\tau$$

$$\left[\begin{array}{l} f(x) = (x^\alpha - x_i^\alpha(\tau))^2 \\ f' = 2(x^\alpha - x_i^\alpha)\left(-\frac{d x_i^\beta}{d\tau}\right)\eta_{\alpha\beta} = \underbrace{-2(x^\alpha - x_i^\alpha) u^\beta(\tau)}_{-2 c r_0(\tau)} \Rightarrow \left|\frac{df}{d\tau}\right|_{\tau_{ret}} = 2 c r_0(\tau_{ret}) \end{array}\right]$$

$$= \frac{q}{2\pi\varepsilon_0 c}\int \frac{\delta(\tau-\tau_{ret})}{2c\, r_0(\tau_{ret})} u^\mu(\tau)\,\Theta(\Delta t)\,d\tau = \frac{q}{4\pi\varepsilon_0}\frac{u^\mu(\tau_{ret})}{c^2 r_0(\tau_{ret})}$$

$$\left[r_0 = -\frac{1}{c} u_\alpha (x^\alpha - x^\alpha_c) = -\frac{1}{c}[\gamma c, \gamma \vec{v}] \begin{bmatrix} \Delta t c \\ \vec{r} \end{bmatrix}_{\tau_{ret}} = \gamma\left(\Delta t - \vec{r}\cdot\vec{v}\right)\Big|_{\tau_{ret}} = \right.$$

$$\left. = \gamma r\left(1 - \frac{\vec{e}\cdot\vec{v}}{c}\right)\Big|_{\tau_{ret}} \right]$$

$c\Delta t|_{\tau_{ret}} = r|_{\tau_{ret}}$
vďaka $\delta((x-x_c(\tau))^2)$

$$\Rightarrow \phi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r\left(1-\frac{\vec{e}\cdot\vec{v}}{c}\right)}\Bigg|_{\tau_{ret}}$$

$$\vec{A} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{v}}{c^2 r\left(1-\frac{\vec{e}\cdot\vec{v}}{c}\right)}\Bigg|_{\tau_{ret}}$$

Elekrodynamika bez zdrojov

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{E} = 0 \qquad \nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\times\nabla\times\vec{E} = \cancel{\nabla(\nabla\cdot\vec{E})}^{\,0} - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\nabla\times\vec{B} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} \quad\Rightarrow\quad \Box\vec{E} = 0$$
$$\nabla\times\nabla\times\vec{B} = \cancel{\nabla(\nabla\cdot\vec{B})}^{\,0} - \nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} \quad\Rightarrow\quad \Box\vec{B} = 0$$

Transverzální vlna

→ skúmame šírenie v smere $\vec{e}_\parallel = konst$ → $r_\parallel = \vec{e}_\parallel\cdot\vec{r}$

→ takúto závislosť predpokladáme v tvare:

$$\vec{E}(t, r_\parallel) = \vec{E}(kr_\parallel - \omega t) = \vec{E}(\psi)$$
$$\vec{B}(t, r_\parallel) = \vec{B}(kr_\parallel - \omega t) = \vec{B}(\psi)$$

↳ potom dostane rovnice

$$\nabla^2\vec{E}(t, r_\parallel) = \vec{E}''(\psi)\cdot k^2 \qquad \partial_t^2 = \vec{E}''\omega^2$$

$$\Rightarrow \Box\vec{E} = \vec{E}''(\psi)\left(k^2 - \frac{\omega^2}{c^2}\right) = 0 \quad \rightsquigarrow \text{automaticky splnené pre } \omega = kc$$

↳ analogicky pre $\vec{B}$

→ $\vec{E}(\psi)$ je profilová funkce $\qquad \psi = kr_\parallel - \omega t = \vec{k}\cdot\vec{r} - \omega t \qquad \vec{k} = k\vec{e}_\parallel$

↳ spätným dosadením do Maxwella:

$$\left.\begin{aligned} \nabla\cdot\vec{E} = 0 &\Rightarrow \vec{k}\cdot\vec{E}' = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0 \\ \nabla\cdot\vec{B} = 0 &\Rightarrow \vec{k}\cdot\vec{B}' = 0 \Rightarrow \vec{k}\cdot\vec{B} = 0 \end{aligned}\right\} \text{kolmé na smer šírenia}$$

$$\left.\begin{aligned} \nabla\times\vec{E} = -\partial_t\vec{B} &\Rightarrow \vec{k}\times\vec{E}' = \omega\vec{B}' \Rightarrow \vec{e}_\parallel\times\vec{E} = c\vec{B} \\ \nabla\times\vec{B} = \frac{1}{c^2}\partial_t\vec{E} &\Rightarrow \vec{k}\times\vec{B}' = -\frac{\omega}{c^2}\vec{E}' \Rightarrow \vec{e}_\parallel\times c\vec{B} = -\vec{E} \end{aligned}\right\} \vec{B}\perp\vec{E}\perp\vec{e}_\parallel$$

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\varepsilon_0 c^2 B^2 = \varepsilon_0 E^2$$
$$\vec{S} = \varepsilon_0 c^2\,\vec{E}\times\vec{B} = \varepsilon_0 c\,\vec{E}\times\vec{e}_\parallel\times\vec{E} = \varepsilon_0 c\left[\vec{e}_\parallel E^2 - \vec{E}(\vec{E}\cdot\vec{e}_\parallel)\right] = \varepsilon_0 c E^2\vec{e}_\parallel = uc\vec{e}_\parallel$$
$$\mathcal{L} \propto E^2 - c^2B^2 = 0 \qquad \tilde{\mathcal{L}} \propto \vec{E}\cdot\vec{B} = 0$$

Monochromatická vlna

↳ profilové fce vieme napísať FŘ: $\vec{E} = \sum a_n \cos \Psi + b_n \sin \Psi$

⇒ budeme skúmať jednoduchú vlnu v tvare $\vec{E} = \vec{E}_0 \sin \Psi$

↳ použitím kpk. čísel môžeme používať

$$\underset{\in \mathbb{C}}{\underbrace{\vec{\mathcal{E}}}} = \vec{\mathcal{E}}_0 e^{i\Psi} \qquad \underset{\in \mathbb{C}}{\underbrace{\vec{\mathcal{B}}}} = \vec{\mathcal{B}}_0 e^{i\Psi}$$

, kde $\vec{E}, \vec{B}$ dostaneme ako reálnu časť

$$\vec{\mathcal{E}}_0 \cdot \vec{e}_\parallel = \vec{\mathcal{B}}_0 \cdot \vec{e}_\parallel = 0$$

$$\vec{e}_\parallel \times \vec{\mathcal{E}}_0 = c\vec{\mathcal{B}}_0 \qquad \vec{e}_\parallel \times c\vec{\mathcal{B}}_0 = -\vec{\mathcal{E}}_0$$

Sférické vlny

↳ Coulombická kalibrace $\nabla \cdot \vec{A} = 0$

$$\nabla^2 \phi = \frac{8}{\varepsilon_0} = 0 \Rightarrow \phi = 0$$

$$\Rightarrow \vec{E} = -\frac{\partial \vec{A}}{\partial t} \qquad \vec{B} = \nabla \times \vec{A}$$

↳ riešime $\Box \vec{A} = 0$

- **Ansatz:**

$\vec{A} = \vec{\mathbb{L}} \Psi$, kde Ψ je skalár (Debyov potenciál) a $\mathbb{L} = -i\vec{r} \times \nabla$

$$\mathbb{L}^2 = \nabla_{S^2}^2 = \frac{1}{\sin\vartheta} \frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial \vartheta}\right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial \varphi^2} \Rightarrow \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\mathbb{L}^2$$

$$\vec{r} \cdot \mathbb{L} = 0 \qquad \Box \mathbb{L} = \mathbb{L} \Box$$

$$\mathbb{L}\vec{r} = 0 \qquad \nabla^2 \mathbb{L} = \mathbb{L} \nabla^2$$

} komutuje

TE - pole ⟵ symetria Maxwellických $\vec{E} \to c\vec{B}$, $c\vec{B} \to -\vec{E}$ ⟶ **TM - pole**

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t} \vec{\mathbb{L}} \Psi^{TE} \qquad c\vec{B} = \frac{\partial}{\partial t} \vec{\mathbb{L}} \Psi^{TM}$$

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \vec{\mathbb{L}} \Psi^{TE} \qquad \frac{1}{c}\vec{E} = \nabla \times \vec{\mathbb{L}} \Psi^{TM}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\vec{r} \cdot \vec{E} = 0 \qquad \vec{r} \cdot \vec{B} = 0$$

$$\vec{r} \cdot \vec{B} = \vec{r} \cdot \nabla \times \vec{\mathbb{L}} \Psi^{TE} = \vec{r} \times \nabla \cdot \vec{\mathbb{L}} \Psi^{TE} = i \mathbb{L}^2 \Psi^{TE} \qquad \vec{r} \cdot \vec{E} = ic\, \mathbb{L}^2 \Psi^{TM}$$

Riešenie

$$\Rightarrow \Box \vec{A} = \Box \vec{\mathbb{L}} \Psi = \vec{\mathbb{L}} \Box \Psi = 0 \Rightarrow \Box \Psi = 0,$$ tj. riešime skalárnu vlnovú rovnicu

$$\Psi = R(r)\, \mathcal{E}(t)\, Y(\vartheta, \varphi)$$

$$\Rightarrow \frac{1}{\psi}\Box\psi = 0 = -\frac{1}{c^2}\underbrace{\frac{\partial^2 \mathcal{E}}{\partial t^2}\frac{1}{\mathcal{E}}}_{-\omega^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\frac{1}{R} + \frac{1}{r^2}\underbrace{\frac{\mathbb{L}^2 Y}{Y}}_{-\ell(\ell+1)} = 0$$

$$\Rightarrow \frac{\partial^2 \mathcal{E}}{\partial t^2} + \omega^2 \mathcal{E} = 0 \quad \Rightarrow \quad \mathcal{E} = e^{i\omega t}$$

$$\Rightarrow -\mathbb{L}^2 Y = \ell(\ell+1) Y \quad \Rightarrow \quad Y = Y_\ell^m(\vartheta, \varphi)$$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + k^2 - \frac{\ell(\ell+1)}{r^2}\right] R = 0$$

sférické Besselovy funkce

$$\Rightarrow R_{k\ell} = \begin{cases} j_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{\ell+\frac{1}{2}}(kr) \\ n_\ell(kr) = \sqrt{\frac{\pi}{2kr}}\, N_{\ell+\frac{1}{2}}(kr) \end{cases}$$

$$j_\ell(\xi) = (-\xi)^\ell \left[\frac{1}{\xi}\frac{d}{d\xi}\right]^\ell \frac{\sin\xi}{\xi} \sim \xi^\ell \quad \xi \ll 1$$

$$n_\ell(\xi) = (-\xi)^\ell \left[\frac{1}{\xi}\frac{d}{d\xi}\right]^\ell \frac{\cos\xi}{\xi} \sim -\frac{1}{\xi^{\ell+1}} \quad \xi \gg 1$$

$$\hookrightarrow \psi(t, r, \vartheta, \varphi) = R_{k\ell}(r)\, Y_\ell^m(\vartheta, \varphi)\, e^{-i\omega t}$$

↳ obecné řešení $\vec{E}, \vec{B}$ má tvar (monochromatické):

$$\frac{1}{c}\vec{E} = \sum_{\ell, m} -a_{\ell m}^{TE} \underbrace{\frac{1}{c}\frac{\partial}{\partial t}}_{\Rightarrow -ik} \vec{\mathbb{L}}\psi_{k,\ell,m}^{TE} + a_{\ell m}^{TM}\, \nabla\times \vec{\mathbb{L}}\psi_{k,\ell,m}^{TM}$$

$$\vec{B} = \sum_{\ell, m} a_{\ell m}^{TE}\, \nabla\times \vec{\mathbb{L}}\psi_{k,\ell,m}^{TE} + a^{TM} \underbrace{\frac{1}{c}\frac{\partial}{\partial t}}_{\Rightarrow ik} \vec{\mathbb{L}}\psi_{k,\ell,m}^{TM}$$

• koeficienty $a_{\ell m}$ dostane ze skalárního součinu $\vec{B}\cdot\vec{r}$, $\vec{E}\cdot\vec{r}$ a integ.:

$$\frac{1}{c}\vec{r}\cdot\vec{E} = \sum a^{TM}\, \vec{r}\cdot\nabla\times\vec{\mathbb{L}}\psi^{TM} = \sum a^{TM}\, i\,\mathbb{L}^2 R_{k\ell} Y_m^\ell e^{-i\omega t} = -i\ell(\ell+1)\sum a^{TM} R_{k\ell} Y_m^\ell e^{-i\omega t}$$

$$\vec{r}\cdot\vec{B} = \sum a^{TE}\, \vec{r}\cdot\nabla\times\vec{\mathbb{L}}\psi^{TE} = \sum a^{TE}\, i\,\mathbb{L}^2 R_{k\ell} Y_m^\ell e^{-i\omega t} = -i\ell(\ell+1)\sum a^{TE} R_{k\ell} Y_m^\ell e^{-i\omega t}$$

$$\Rightarrow a^{TM} R_{k\ell} = \frac{i}{\ell(\ell+1)} e^{i\omega t} \int \frac{1}{c}\vec{r}\cdot\vec{E}\, Y_{\ell m}^*\, d^2\Omega$$

$$a^{TE} R_{k\ell} = \frac{i}{\ell(\ell+1)} e^{i\omega t} \int \vec{r}\cdot\vec{B}\, Y_{\ell m}^*\, d^2\Omega$$

$\vec{E}, \vec{B} \propto e^{-i\omega t}$ → už harmonické