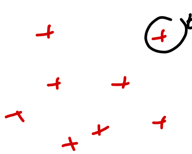
$a = 1, 2, \dots, n$

$$\vec F_a = \sum_b \frac{q_a q_b}{4\pi\varepsilon_0} \frac{\vec x_a - \vec x_b}{|x_a - x_b|^3}$$

- Coulomb
- princip superpozice

- n je hodne veliký
- pohodlnejší počítat integrály ale integrály $\sum_i \cos nx$ vs. $\int \cos x \, dx$



chytrý náboj – vie o polohe všetkých ostatných nábojov

Pole:
1. vytvárajú ho náboje
2. náboje pole cítí

$$\vec F_a = q_a \vec E(\vec x_a)$$

$$\vec E(\vec x) = \sum_b \frac{q_b}{4\pi\varepsilon_0} \frac{\vec x - \vec x_b}{|\vec x - x_b|^3}, \quad \text{ale} \quad E(x_a) = \text{"}0\text{"} + \sum_{b \neq a} \frac{1}{4\pi\varepsilon_0} \frac{\vec x_a - \vec x_b}{|x_a - x_b|^3}$$

→ aby toto platilo náboj nepôsobí sám na seba

3. teorie určuje jak se pole "rozprostírá"
4. vše lokálne

⇒ jazyk dif. rovnic

Typy pole:
1. fyzikální → skalární, vektorová, tenzorová
   → veličina určená ve prostorových a čas. souřadnicích
   $\vec E(\vec x, t)$, $\phi(\vec x)$, ...
2. matematická
   $f(\vec x, t)$

Pozn: vybereme konkrétní souř. systém

$$F_{\mu\nu} \mapsto V^{\alpha} \to \vec{E}, \vec{B}$$

$\underbrace{f, g,}_{\text{skaláry}}$ $\underbrace{\vec{A}, \vec{B}}_{\text{vektory}}$ $\frac{\partial}{\partial t}$ ... skalární $\frac{\partial}{\partial x^i} = \nabla$ ... vektor

s: $f, f \cdot g, \vec{A} \cdot \vec{B}, \partial_t f, \nabla \cdot \vec{A} = \partial_i A_i,$

v: $\vec{A}, f\vec{A}, \vec{A} + \vec{B}, \vec{A} \times \vec{B}, \underbrace{\nabla f}_{\text{grad}} \text{"="} \partial_i f, \nabla \times \vec{A}$

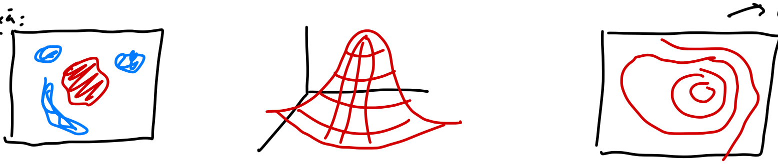
∂_t ... dynamika

∇ .... z místa na místo se pole mění

Skalární pole

$f(\vec{x})$, příklad: elstat. potenciál $\phi(\vec{x})$

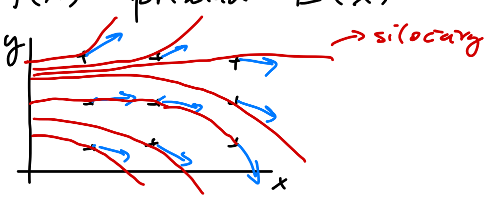
vizualizace:



Př: $f = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{1}{r}$

Vektorové pole

$\vec{A}(\vec{x})$ – příklad: $\vec{E}(\vec{x})$



vektorové čáry (siločáry)

$\vec{x}(s)$ ... křivka

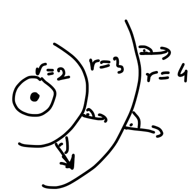
↳ s je parametr

$$\frac{d\vec{x}(s)}{ds} = \underbrace{\lambda(s)}_{\text{vhodně zvolená fce}} \vec{E}(\vec{x}(s))$$ → rovnice siločáry

E_i , $i = 1, 2, 3$

$$\vec{E} = \left(\frac{q}{4\pi\varepsilon_0} \frac{x}{(\sqrt{x^2+y^2+z^2})^3}, \frac{q}{4\pi\varepsilon_0} \frac{y}{(\sqrt{x^2+y^2+z^2})^3}, \frac{q}{4\pi\varepsilon_0} \frac{z}{(\sqrt{x^2+y^2+z^2})^3} \right)$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{\vec{x}}{|\vec{x}|^3} = \frac{q}{4\pi\varepsilon_0} \frac{\vec{e}_r}{r^2} \qquad \vec{e}_r = \frac{\vec{x}}{|\vec{x}|}$$



$$r = \sqrt{x^2+y^2+z^2} \qquad \nabla r = [\partial_x r, \partial_y r, \partial_z r] = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r} = \vec{e}_r$$

$$\nabla r = \vec{e}_r$$

$$\nabla f(r) = \frac{df}{dr}\nabla r = f'\vec{e}_r$$

$$\nabla \frac{1}{r} = -\frac{1}{r^2}\vec{e}_r$$

$$\nabla r^\alpha = \alpha r^{\alpha-1}\vec{e}_r$$

Nabla ve výrazech

$$\nabla(fg) = (\nabla f)g + f(\nabla g)$$
$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot\nabla f$$

Elekrostatika

- $\vec{j} = 0$ bez proudů
- $\vec{B} = 0$ bez magnetov
- $\partial_t = 0$ nič nezávisí na čase

$$\boxed{\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon} \\ \nabla \times \vec{E} &= 0 \end{aligned}}$$

Potíž:

$$\vec{E} = \sum_b \frac{q_b}{4\pi\varepsilon_0} \frac{\vec{x} - \vec{x}_b}{|\vec{x} - \vec{x}_a|^3}$$

pro 1 náboj

$$\vec{E}(\vec{x}) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{x}}{|\vec{x}|^3}$$

$\forall |x| \neq 0$

$$\nabla \cdot \vec{E} = \frac{q}{4\pi\varepsilon_0}\left(\left(\nabla \cdot \frac{1}{|x|^3}\right) \cdot \vec{x} + \frac{1}{|x|^3} \nabla \cdot \vec{x}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{-3}{|x|^4} \frac{\vec{x}}{|x|} \cdot \vec{x} + \frac{3}{|x|^3}\right) = 0$$

Zavedení potenciálu

$$\nabla \times (\nabla f) = 0 \quad \Rightarrow \quad \exists \phi : \quad \boxed{\vec{E} = -\nabla \phi} \quad \rightarrow \text{pro elstat}$$

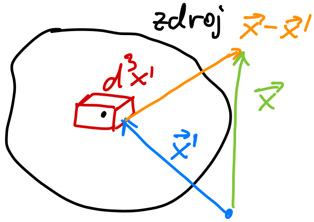
$$\Rightarrow \boxed{\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}} \quad \text{Poisson}$$

$$\vec{E}(\vec{x}) = \sum_a \underbrace{\frac{q_a}{4\pi\varepsilon_0} \frac{1}{|\vec{x} - \vec{x}_a|}}_{f'} \underbrace{\frac{\vec{x} - \vec{x}_a}{|\vec{x} - \vec{x}_a|^2}}_{\frac{\vec{r}}{r}}$$

$$\nabla f(r) = f' \frac{\vec{r}}{r} \quad \Rightarrow \quad \phi = \sum_a \frac{q_a}{4\pi\varepsilon_0} \frac{1}{|x - x_a|}$$

Hustoty nábojů

$$dq = \underbrace{\rho(\vec{x}')}_{\text{obj. náboj. hust.}} d^3x' = \underbrace{\sigma(\vec{x}')}_{\text{plošná náboj. hust.}} dS' = \underbrace{\lambda(\vec{x}')}_{\text{lineární náboj. hust.}} dl'$$



s limitou $\sum_a \Delta q_a \longrightarrow \int dq$

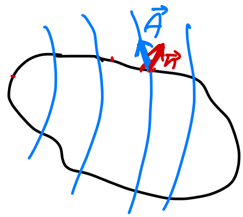
$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\,\rho(\vec{x}')\,d^3x'}$$

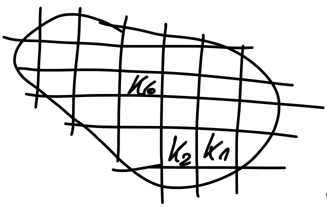
$$\boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'}$$

potiž $\nabla\cdot\int\cdots \neq \int\nabla\cdot$, lebo nesplňuje podm. na prehodenie (∞ pre $\vec{x}=\vec{x}'$)
Tak spočítame, kde konv. vyjde 0, ale to platí mimo $\vec{x}=\vec{x}'$, pričom ρ je nenulové práve v $\vec{x}'$

Gaussova veta

• mat.: $\displaystyle\oint_{\partial\Omega} \vec{A}\cdot d\vec{S} = \int_\Omega \nabla\cdot\vec{A}\; d^3x$





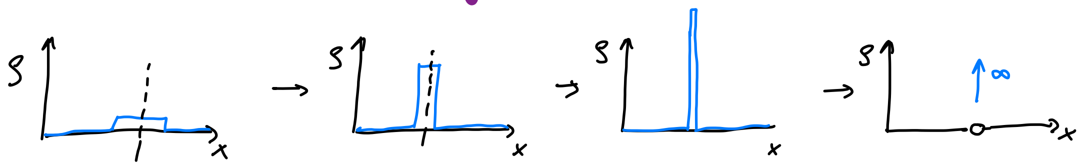
$$\oint_{\partial\Omega} \vec{A}\cdot d\vec{S} = \sum_a \overbrace{\oint_{\partial K_a} \vec{A}\cdot d\vec{S}}^{d(\vec{x}_a)V_{k_a}} \rightarrow \sum_a d(x_a)\,\Delta V_a \rightarrow \int \nabla\cdot\vec{A}\; dV$$

$$\boxed{\nabla\cdot\vec{A} \overset{(1)}{:=} \lim_{\Delta V\to 0^*} \frac{1}{\Delta V}\oint_{\partial V} \vec{A}\cdot d\vec{S}}$$

↳ hviezdička znamená že objem limitime do bodu (nie plochy/čiary)

• fyz: $\boxed{\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$

Hustota bodového náboje



$$Q = \int \rho \, dV$$

- def. symbol $\delta^{(3)}(x)$: $\int_{\mathbb{R}^3} \delta^{(3)}(x)\, d^3x = 1$, $\delta(\vec{x}) = 0 \quad \forall \vec{x} \neq 0$

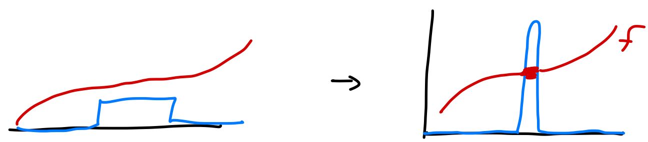
- nábojová hust. bodového náboje v místě $\vec{x}\,'$ o velikosti q':

$$\boxed{\rho(\vec{x}) = q' \delta^{(3)}(\vec{x} - \vec{x}\,')}$$

$$\int_{\Omega} \delta^{(3)}(\vec{x} - \vec{x}\,')\, d^3x = \begin{cases} 0 & x' \notin \Omega \setminus \partial\Omega \\ 1 & x' \in \Omega \setminus \partial\Omega \\ \text{nedef} & x' \in \partial\Omega \end{cases}$$

$f\delta$ →

↑ hezká fce



$$f(\vec{x})\,\delta^{(3)}(\vec{x} - \vec{x}\,') \equiv f(\vec{x}\,')\,\delta^{(3)}(\vec{x} - \vec{x}\,')$$

Důsledek:

$$\boxed{\int f(x)\, \delta^{(3)}(\vec{x} - \vec{x}\,')\, d^3x = f(\vec{x}\,')}$$

distribuce ≡ zobecněná fce ∋ hezké fce, $\delta(\vec{x})$, $f\delta$, $d_1 + d_2$

- při integraci platí – věta o substituci, per partes, Gauss, Stokes
- rovnost dvou distribucí se testuje pod integračním znaménkem

$$\int \psi(x)\,\delta^3(\vec{x})\,d^3x \neq \int \psi(x)\, 2\delta^3(\vec{x})\,d^3x$$
$$\psi(0) \qquad \neq 2\psi(0) \qquad \Rightarrow \delta^3(\vec{x}) \neq 2\delta^3(\vec{x})$$

- prostor zobecnejších funkcí je úplnejší, $\exists$ limity

$$\rho(\vec{x}) = \sigma\,\delta(\vec{x} - \text{povrch})$$

$$\nabla \cdot \overbrace{\int \frac{1}{4\pi\varepsilon_0} \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}')\,d^3x'}^{\vec{E}} \overset{?}{=} \frac{1}{\varepsilon_0}\int \delta^3(\vec{x}-\vec{x}')\rho(\vec{x}')\,d^3x' = \frac{\rho(\vec{x})}{\varepsilon_0}$$

?

$$\nabla \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\frac{1}{4\pi} = \delta^3(\vec{x}-\vec{x}')$$

treba overiť

ψ hezké... $\psi(\infty) = 0$

ĽS:

$$\int \psi(x')\,\nabla\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)\frac{1}{4\pi}\,d^3x = \int \nabla\cdot\left[\psi(x')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\frac{1}{4\pi}\right]d^3x - \int \frac{1}{4\pi}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\nabla\psi\,d^3x =$$

$$= \cancel{\oint \psi(\vec{x}')\frac{1}{4\pi}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\cdot d\vec{S}} \xrightarrow{} 0 \text{ pro } \vec{x}\to\infty \qquad - \int \frac{1}{4\pi}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\cdot\nabla\psi\,d^3x =$$

trans. súradníc
$\vec{x}-\vec{x}' = \vec{r}$
$d^3x = r^2\,d\Omega\,dr$
pros. f. chol

objemová integrace je $\mathbb{R}^3$, teda integ. plocha obklopuje $\mathbb{R}^3$ a keďže $\psi(\infty)=0$ zmizne

$$= -\int \frac{1}{4\pi}\frac{\vec{r}}{r^3}\cdot\nabla\psi\; r^2\,d\Omega\,dr = -\frac{1}{4\pi}\int\left(r\frac{\partial}{\partial r}\psi\right)\frac{1}{r^3}\,r^2\,d\Omega\,dr =$$

$$= -\frac{1}{4\pi}\int_{\langle 4\pi\rangle}\left[\int_0^\infty \frac{\partial\psi}{\partial r}\,dr\right]d\Omega = -\frac{1}{4\pi}\int_{\langle 4\pi\rangle}\left[\cancelto{0}{\psi(\infty)} - \psi(0)\right]d\Omega = \psi(0) = \psi(\vec{x}')$$

PS: $\psi(\vec{x})\,\delta^3(\vec{x}-\vec{x}') = \psi(\vec{x}') \qquad \Rightarrow \quad$ ĽS = PS

Pre potenciál?

$$\phi(\vec{x}) = \int \frac{\rho(\vec{x}')}{4\pi\varepsilon_0|\vec{x}-\vec{x}'|}\,d^3\vec{x}' \quad \to \text{platí } \nabla^2\phi = -\frac{\rho}{\varepsilon_0} \;??$$

áno

$$\boxed{\nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}')}$$

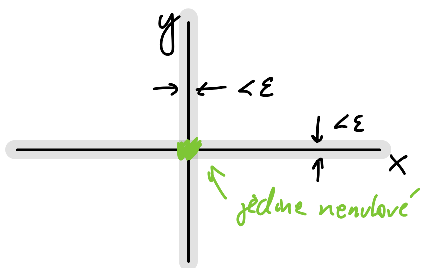
$\nabla^2 \frac{1}{r}$ "=" 0 $\;\to$ neplatí pro $r=0$

$$\delta^3(\vec{x}) \begin{cases} |\vec{x}| \neq 0 & \text{“}\rho\text{”} = 0 \\ \vec{x} = 0 & \text{“}\rho\text{”} = \text{“}\infty\text{”} \end{cases}$$

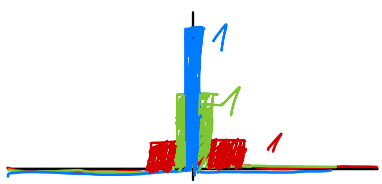
$$\int \delta^3(\vec{x})\, d^3x = 1$$

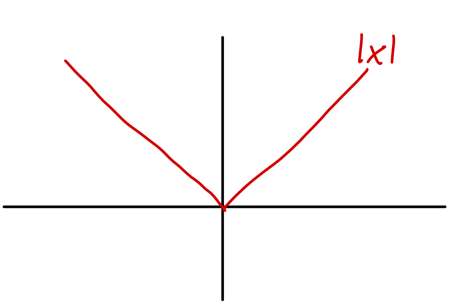
$$\delta^3(\vec{x}) = \delta(x)\,\delta(y)\,\delta(z)$$

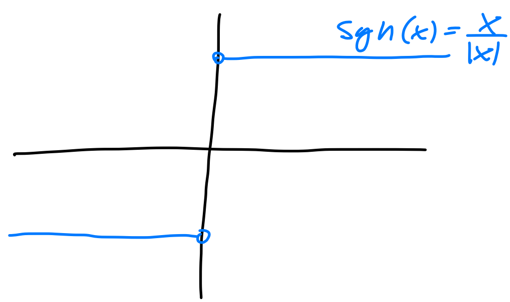
$$\int_{\mathbb{R}^3} \delta^3(\vec{x})\, d^3x = \int_{\mathbb{R}} \delta(x)\,dx \int_{\mathbb{R}} \delta(y)\,dy \int_{\mathbb{R}} \delta(z)\,dz = 1 \cdot 1 \cdot 1 = 1$$



“graf $\delta(x)$”

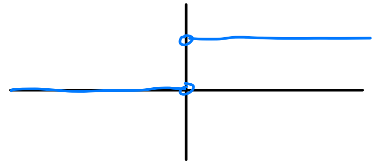






$$\frac{d}{dx}|x| = \mathrm{sgn}(x) \qquad \int^x \mathrm{sgn}(x')\,dx' = |x| + C$$

$$\int_{-\infty}^{x}\delta(x')\,dx = \begin{cases} 0 & x<0 \\ 1 & x>1 \end{cases}$$



$$\Rightarrow \boxed{\frac{d}{dx}\,\mathrm{sgn}(x) = 2\delta(x)}$$

Dk: $\int \psi(x)\frac{d}{dx}\,\mathrm{sgn}(x) \overset{?}{=} \int \psi(x)\, 2\delta(x)\,dx = 2\psi(0)$

$$\|$$

$$[\psi(x)\,\mathrm{sgn}(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\psi'(x)\,\mathrm{sgn}(x) \overset{\psi(\mp\infty)=0}{=} \left(-\int_{-\infty}^{0} - \int_{0}^{\infty}\right)\psi'(x)\,\mathrm{sgn}(x) =$$

$$= [\psi]_{-\infty}^{0} - [\psi]_{0}^{\infty} = 2\psi(0) \qquad \square$$

3D bodový náboj (0D v 3D)

$$\nabla^2 \frac{1}{r} = -4\pi\,\delta^3(\vec{r})$$

- potenciál bodového náboje v počátku: $\phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$

$$\nabla^2\phi = \frac{Q}{4\pi\varepsilon_0}\nabla^2\frac{1}{r} = -\frac{Q}{\varepsilon_0}\,\delta^3(\vec{r}) \Rightarrow \boxed{\rho(\vec{r}) = Q\,\delta^3(\vec{r})}$$

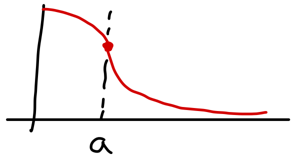
1) Aplikace Poissonovy rovnice

2) Použití Gaussovy věty

$$\phi \longmapsto E = -\nabla\phi \quad \Rightarrow \quad \vec{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

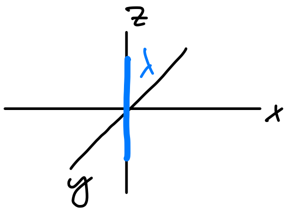
$$\oint_{S^3} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0} \qquad \forall r>0$$

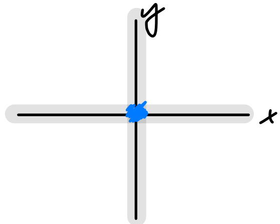
3) $\phi = \frac{Q}{4\pi\varepsilon_0}\cdot\begin{cases} r>a: & \frac{1}{r} \\ r\le a & \frac{3a^2-r^2}{2a^3}\end{cases}$



$$\nabla^2\phi = \frac{1}{r}(r\phi)'' = \frac{Q}{4\pi\varepsilon_0}\begin{cases} r>a: & 0 \\ r\le a: & -\frac{3}{a^3}\end{cases} = -\frac{1}{\varepsilon_0}\begin{cases} r>a: & 0 \\ r\le a: & \frac{Q}{\frac{4}{3}\pi a^3}\end{cases}$$

3D lineární náboj (1D v 3D)





$$\nabla^2 \log\left(\sqrt{x^2+y^2}\right) = 2\pi\,\delta(x)\delta(y)$$

λ... lineární nábojová hustota

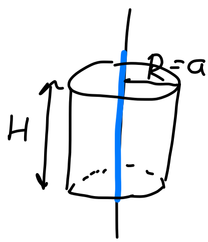
$$dQ = \lambda\, dz$$

$$\Delta \frac{\lambda}{2\pi}\log\sqrt{x^2+y^2} = -\frac{1}{\varepsilon_0}\left[-\lambda(z)\delta(x)\delta(y)\right]$$

$$\boxed{\phi = -\frac{\lambda}{2\pi\varepsilon_0}\ln R}$$

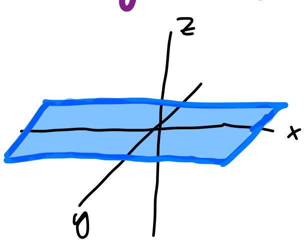
$$\nabla\phi = -\frac{\rho}{\varepsilon_0}$$

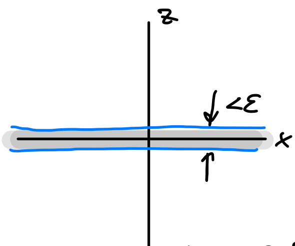
2) Gauss:



$E_R = -\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{R}$ $\qquad Q = -\lambda H$

Plošný náboj v 3D (2D v 3D)



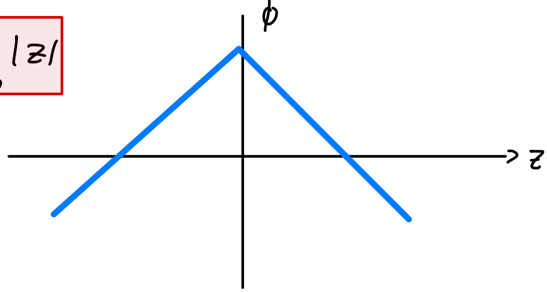


$\Delta\phi = -\frac{\rho}{\varepsilon_0}$ $\qquad \rho = \sigma\delta(z)$

$$\frac{d^2}{dz^2}|z| = \frac{d}{dz}\,\mathrm{sgn}(z) = 2\delta(z)$$

$$\Delta|z| = 2\delta(z) \to \Delta\Big(\underbrace{\frac{\sigma}{-2\varepsilon_0}|z|}_{\phi}\Big) = -\frac{1}{\varepsilon_0}\overbrace{\sigma\delta(z)}^{\rho}$$

$$\boxed{\phi = -\frac{\sigma}{2\varepsilon_0}|z|}$$



→ pro $z=0$ není $\phi=\infty$ ako v prípade 0D, 1D

↳ dále má nespojitú deriváci

Singulární chování potenciálu

- bodové a lin. náboje $\longmapsto \phi = \pm\infty$
- plošné náboje $\longmapsto \nabla\phi$ nespojitý

$$\lim_{\varepsilon\to 0} f(\vec{x}+\vec{n}\varepsilon) - f(\vec{x}-\vec{n}\varepsilon) =: [f]$$ → skok fce

$$\boxed{\vec{n}\cdot[\vec{E}] = \frac{\sigma}{\varepsilon_0}}$$

- vždy ϕ je spojité

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\sum_a \frac{q_a}{|\vec{x}-\vec{x}_a|}$$

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{d^3\vec{x}'\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} \qquad \tilde{\rho} = \frac{\rho}{\varepsilon_0}$$

$$\phi(\vec{x}) = \int G(\vec{x},\vec{x}')\,\tilde{\rho}(\vec{x}')\,d^3\vec{x}'$$

$$G(\vec{x},\vec{x}') = \frac{1}{4\pi}\frac{1}{|\vec{x}-\vec{x}'|}$$

$$\boxed{\nabla^2 G(\vec{x},\vec{x}') = -\delta^3(\vec{x}-\vec{x}')}$$ → Greenova úloha

↳ Greenova funkce

Greenova úloha

- analogie: $A\vec{y} = \vec{b}$ $\qquad A_{ij}\, y_j = b_i$

$$A\vec{g}_k = \vec{e}_k \qquad A_{ij}(g_k)_j = \delta_{ki}$$

$$\vec{b} = \sum_k b_k \vec{e}_k \qquad b_i = \sum_k b_k \delta_{ki}$$

$$A\left(\sum_k b_k \vec{g}_k\right) = \sum_k b_k \vec{e}_k \qquad A_{ij}\left(\sum_k b_k (g_k)_j\right) = b_i$$

$$\Rightarrow \vec{y} = \sum b_k \vec{g}_k \qquad y_i = \sum_k b_k (g_k)_i$$

→ analogicky:

$$\underbrace{-\nabla^2}_{\text{lin. operator}} \phi = \tilde{\rho} \qquad \tilde{\rho}(\vec{x}) = \int \tilde{\rho}(\vec{x}')\, \delta^3(\vec{x} - \vec{x}')\, d^3x$$

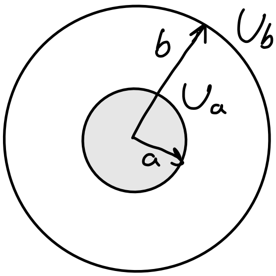
$$-\nabla^2 G(\vec{x}, \vec{x}') = \delta^3(\vec{x} - \vec{x}')$$

$$-\nabla^2 \int G(\vec{x}, \vec{x}')\, \tilde{\rho}(\vec{x}')\, d^3x' = \tilde{\rho}(x)$$

$$\boxed{\begin{aligned} \phi(\vec{x}) &= \int G(\vec{x}, \vec{x}')\, \tilde{\rho}(\vec{x}')\, d^3x' \\ -\nabla^2 G(\vec{x}, \vec{x}') &= \delta^3(\vec{x} - \vec{x}') \end{aligned}}$$

Poissonova rovnice jako PDR s okraj. podm. (příklad)

$\Delta \phi = 0$ Laplaceova rovnice

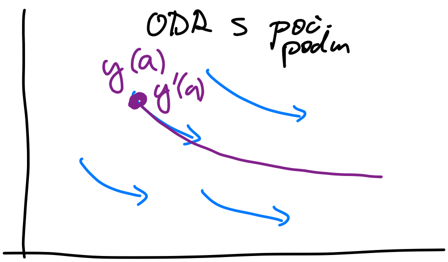


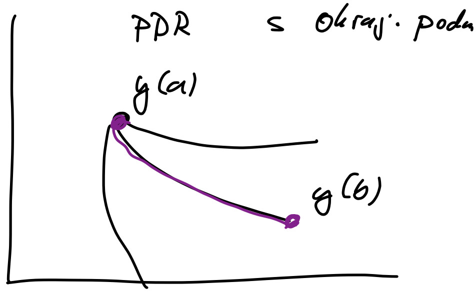
symetrie problému ⇒ sférické

⇒ $\phi(\vec{r}) = \phi(r)$

$$\text{PDR} \xrightarrow{\text{symetrie}} \text{ODR}$$

$$\frac{1}{r}(r\phi)'' = 0$$
$$(r\phi)'' = 0$$
$$(r\phi)' = A$$
$$r\phi = Ar + B$$
$$\phi = A + \frac{B}{r}$$



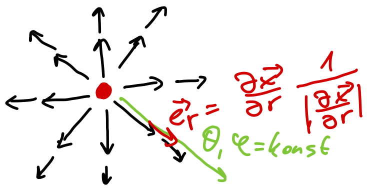


$$U_a = A + \frac{B}{a}$$
$$U_b = A + \frac{B}{b}$$

$$\Rightarrow aU_a - bU_b = (a-b)A \qquad U_a - U_b = B\left(\frac{1}{a} - \frac{1}{b}\right)$$

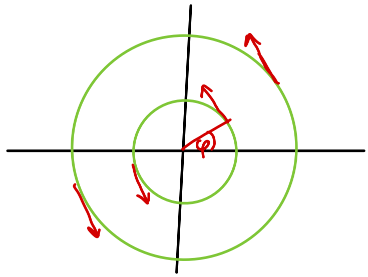
$$A = \frac{bU_b - aU_a}{b-a}, \quad B = \frac{U_a - U_b}{1/a - 1/b} = ab\frac{U_a - U_b}{b-a}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\vec{e}_r \quad , \quad r = \sqrt{x^2+y^2+z^2} \qquad \vec{e}_r = \frac{\vec{x}}{|\vec{x}|}$$



Válcové suradnice R, φ, z

$$\vec{e}_\varphi = \frac{\partial \vec{x}}{\partial \varphi}\frac{1}{\left|\frac{\partial \vec{x}}{\partial \varphi}\right|}$$

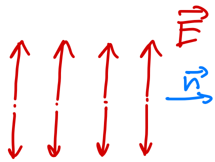


Symetrie

Translační

$\phi(\vec{x} + s\vec{n}) = \phi(\vec{x}) \quad \forall s \in \mathbb{R}$

$\vec{E}(\vec{x} + s\vec{n}) = \vec{E}(\vec{x})$



- adaptace souřadnic $\quad \vec{n} = \vec{e}_z \quad \Rightarrow \quad \phi = \phi(x, y, \cancel{z})$, $\quad \vec{E} = \vec{E}(x, y, \cancel{z})$

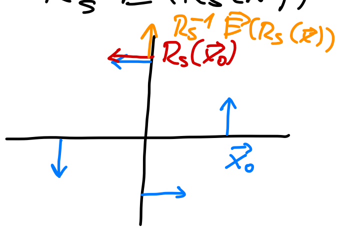
↳ $\nabla \phi = 0$ se zjednoduší: $\partial_{xx}\phi + \partial_{yy}\phi = 0$

Axiální

$\phi(R_s(\vec{x})) = \phi(\vec{x})$

po otočení o úhel s okolo osy

$$\hat{R}_s^{-1}\vec{E}(R_s(\vec{x})) = \vec{E}(\vec{x})$$



- adaptace souřadnic: – válcové / sférické souř
  – osa rotace = rotace z

- válcové souřadnice

$$\phi = \phi(R, \cancel{\varphi}, z)$$

$$\vec{E} = E_R(R,z)\vec{e}_R + E_\varphi(R,z)\vec{e}_\varphi + E_z(R,z)\vec{e}_z$$

- sférické souřadnice

$$\vec{E} = E_r(r,\theta)\vec{e}_r + E_\theta(r,\theta)\vec{e}_\theta + E_\phi(r,\theta)\vec{e}_\varphi$$

Sférické

$$\phi(r)$$

$$\vec{E} = E_r(r)\vec{e}_r$$

Souřadnice

Kartézské

x, y, z $\qquad \vec{e}_x, \vec{e}_y, \vec{e}_z$ $\qquad \vec{e}_i \times \vec{e}_j = \varepsilon_{ijk}\vec{e}_k$

konstantní vek. pole

Válcové

R, φ, z $\qquad \vec{e}_R, \vec{e}_\varphi, \vec{e}_z$

$$x = R\cos\varphi$$

$$y = R\sin\varphi$$

$$z = z$$

$$dl^2 = dR^2 + R^2 d\varphi^2 + dz^2$$

Sférické

r, φ, ϑ

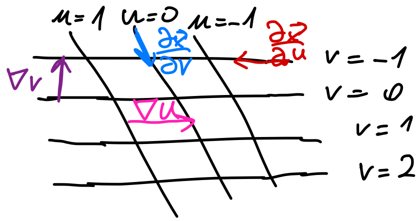
$$x = r\cos\varphi \sin\vartheta$$

$$y = r\sin\varphi\sin\vartheta$$

$$z = r\cos\vartheta$$

$$dl^2 = dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta\, d\varphi^2$$

Ortogonální křivočaré souřadnice



3D: $x, y, z \longrightarrow q_1, q_2, q_3$

$$d\vec{x} = \frac{\partial \vec{x}}{\partial q_i} dq_i \quad \rightarrow \quad d\vec{x} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \\ dq_3 \end{pmatrix}$$

$$d\vec{q} = \frac{\partial \vec{q}}{\partial x_i} dx_i \quad \rightarrow \quad d\vec{q} = \begin{pmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} & \frac{\partial q_1}{z} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} & \frac{\partial q_2}{\partial z} \\ \frac{\partial q_3}{\partial x} & \frac{\partial q_3}{\partial y} & \frac{\partial q_3}{\partial z} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$\left(\vec{V}_1 \ \vec{V}_2 \ \vec{V}_3\right)^{-1} = \begin{pmatrix} \frac{1}{|V_1|^2} \vec{V}_1 \\ \frac{1}{|V_2|^2} \vec{V}_2 \\ \frac{1}{|V_3|^2} \vec{V}_3 \end{pmatrix}$$

OG. sour.

OG. sour

Lameho koef.

$$\boxed{\frac{\partial \vec{x}}{\partial q_i} = h_i \vec{e}_i} \qquad |\vec{e}_i| = 1$$

$$d\vec{l} = h_1 \vec{e}_1 dq_1 + h_2 \vec{e}_2 dq_2 + h_3 \vec{e}_3 dq_3$$

$$dl^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$$

$$\nabla q = \frac{1}{h_i} \vec{e}_i$$

karteźske : $h_1 = h_2 = h_3 = 1$

válcové : $h_1 = h_3 = 1 \quad h_2 = r$

sférické : $h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\vartheta$

$$\vec{e}_r = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)$$

$$\vec{e}_\varphi = h_\varphi \nabla\varphi = r\sin\theta\ \nabla \operatorname{arctg}\left(\frac{y}{x}\right) = r\sin\theta\left[\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2},\ \frac{1}{x},0\right)\right] =$$

$$= r\sin\theta\left[\frac{1}{x^2+y^2}(-y, x, 0)\right] = \frac{\sqrt{x^2+y^2}}{x^2+y^2}(-y, x, 0) = \frac{1}{\sqrt{x^2+y^2}}(-y, x, 0)$$

$$= (-\sin\vartheta\sin\varphi,\ \sin\vartheta\cos\varphi,\ 0)$$

Gradient

$$\vec{e}_z = A\vec{e}_r + B\vec{e}_\theta + C\vec{e}_\varphi$$

$$A = \vec{e}_r \cdot \vec{e}_z = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)\cdot(0,0,1) = \frac{z}{r} = \cos\theta$$

$$B = \vec{e}_\theta \cdot \vec{e}_z = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)(0,0,1) = -\sin\theta$$

$$C = \vec{e}_\varphi \cdot \vec{e}_z = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta)\cdot(0,0,1) = \cos\theta$$

$$(\nabla f)_i = \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial q_j}\frac{\partial q_j}{\partial x_i} = \frac{\partial f}{\partial q_j}\frac{1}{h_j}(\vec{e}_j)_i$$

$$\boxed{\nabla f = \frac{1}{h_j}\frac{\partial f}{\partial q_j}\vec{e}_j} \Rightarrow \boxed{\nabla = \frac{1}{h_j}\frac{\partial}{\partial q_j}\vec{e}_j}$$

Divergence

$$\nabla\cdot\vec{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint \vec{A}\cdot d\vec{S} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\left(\int_{S_1^+} + \int_{S_1^-} + \dots + \int_{S_3^-}\right)\vec{A}\cdot d\vec{S}$$

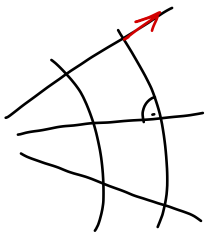
$$\int_{S_1^+} \vec{A}\cdot d\vec{S} = \int A_1 dS_1 = \int A_1 h_2 h_3 \, dq_2 \, dq_3 = [A_1 h_2 h_3]_{q_1+\frac{\Delta q_1}{2}} \Delta q_2 \Delta q_3$$

$$\int_{S_1^-} \vec{A}\cdot d\vec{S} = \int \vec{A}\left(-\vec{e}_1 dS_1\right) = -[A_1 h_2 h_3]_{q_1-\frac{\Delta q_1}{2}} \Delta q_2 \, q_3$$

$$\Delta V = h_1 h_2 h_3 \, \Delta q_1 \Delta q_2 \Delta q_3$$

$$\frac{\int_{S_1^+} + \int_{S_1^-}}{\Delta V} = \frac{A_1 h_2 h_3 [q_1 + \frac{\Delta q_1}{2}] - A_1 h_2 h_3 [q_1 - \frac{\Delta q_1}{2}] \cancel{\Delta q_2 \Delta q_3}}{h_1 h_2 h_3 \, \Delta q_1 \cancel{\Delta q_2 \Delta q_3}} =$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_1 h_3) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right]$$



$$\frac{\partial \vec{x}(q_1, q_2, q_3)}{\partial q_1} = h_1 \vec{e}_1$$

$$d\vec{\ell} = h_1 \vec{e}_1 dq_1 + h_2 \vec{e}_2 dq_2 + h_3 \vec{e}_3 dq_3$$

$$\frac{d\vec{x}(s)}{ds} = \lambda \vec{E}(x(s))$$

$$h_1 \vec{e}_1 \frac{dq_1}{ds} + h_2 \vec{e}_2 \frac{dq_2}{ds} + h_3 \vec{e}_3 \frac{dq_3}{ds} = \lambda \vec{E}_1 \vec{e}_1 + \lambda \vec{E}_2 \vec{e}_2 + \lambda \vec{E}_3 \vec{e}_3$$

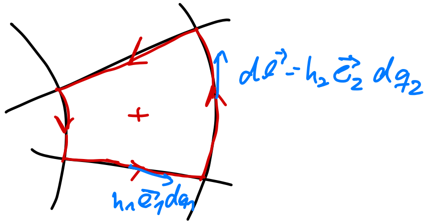
$$h_1 \frac{dq_1}{ds} = \lambda E_1 \qquad h_2 \frac{dq_2}{ds} = \lambda E_1 \qquad h_3 \frac{dq_3}{ds} = \lambda E_1$$

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\vec{e}_3$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_1 h_3) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right]$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{\partial f}{\partial q_1}\frac{h_1 h_2 h_3}{h_1^2}\right) + \frac{\partial}{\partial q_2}\left(\frac{\partial f}{\partial q_2}\frac{h_1 h_2 h_3}{h_2^2}\right) + \frac{\partial}{\partial q_3}\left(\frac{\partial f}{\partial q_3}\frac{h_1 h_2 h_3}{h_3^2}\right)\right]$$

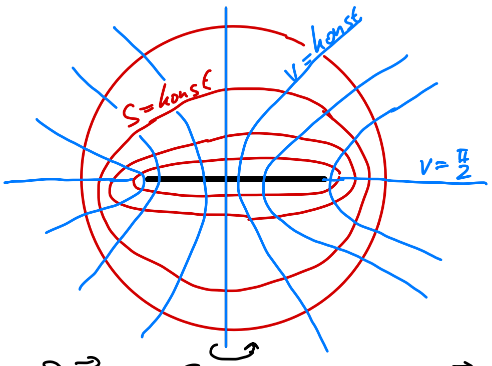
$$\oint_{\partial S} \vec{A}\cdot d\vec{\ell} = \int_S \nabla\times\vec{A}\cdot d\vec{S} \Rightarrow \vec{n}\cdot\nabla\times\vec{A} = \lim_{\Delta S\to 0} \frac{1}{\Delta S}\oint_{\partial S} \vec{A}\cdot d\vec{\ell}$$



$$\frac{A_1 h_2\left(q_2 - \frac{\Delta q_2}{2}\right) - A h_1\left(q_2 + \frac{\Delta q_2}{2}\right)}{h_1 h_2\, dq_2\, dq_3}$$

$$\boxed{\nabla\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\vec{e}_1 & h_2\vec{e}_2 & h_3\vec{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}}$$

Vodivá minca (disk)



$$x = \sqrt{s^2+a^2}\,\sin v\cos\varphi$$
$$y = \sqrt{s^2+a^2}\,\sin v\sin\varphi$$
$$z = s\cos v$$

$$\frac{\partial\vec{x}}{\partial s} = \frac{s}{\sqrt{s^2+a^2}}\sin v\cos\varphi\,\vec{e}_x + \frac{s}{\sqrt{s^2+a^2}}\sin v\sin\varphi\,\vec{e}_y + \cos v\,\vec{e}_z$$

$$\frac{\partial\vec{x}}{\partial v} = \sqrt{s^2+a^2}\cos v\cos\varphi\,\vec{e}_x + \sqrt{s^2+a^2}\cos v\sin\varphi\,\vec{e}_y - s\sin v\,\vec{e}_z$$

$$\frac{\partial\vec{x}}{\partial\varphi} = \sqrt{s^2+a^2}\cos v(-\sin\varphi)\vec{e}_x + \sqrt{s^2+a^2}\cos v\cos\varphi\,\vec{e}_y$$

$$h_s^2 = \left(\frac{\partial \vec{x}}{\partial s}\right)^2 = \frac{s^2}{s^2+a^2}\sin^2 v + \cos^2 v = \frac{s^2+a^2\cos^2 v}{s^2+a^2}$$

$$h_s = \sqrt{\frac{s^2+a^2\cos^2 v}{s^2+a^2}}$$

$$h_v = \sqrt{s^2+a^2\cos^2 v} \qquad g = \begin{pmatrix} \sqrt{\frac{s^2+a^2\cos^2 v}{s^2+a^2}} & & \\ & \sqrt{s^2+a^2\cos^2 v} & \\ & & \sqrt{s^2+a^2}\sin v \end{pmatrix}$$

$$h_\varphi = \sqrt{s^2+a^2}\sin v$$

$$\nabla^2 \phi(s) = 0 \qquad s>0$$

$$\Downarrow$$

$$\cancel{\frac{1}{h_s h_v h_\varphi}} \frac{\partial}{\partial s}\left(\frac{h_v h_\varphi}{h_s}\frac{\partial}{\partial s}\phi(s)\right) = 0$$

$$\frac{h_v h_\varphi}{h_s} = (s^2+a^2)\sin v \quad \Rightarrow \quad \left[(s^2+a^2)\phi'(s)\right]' = 0$$

$$\Rightarrow \phi'(s) = \frac{\tilde{A}}{s^2+a^2} \Rightarrow \phi(s) = A\left(\operatorname{arctg}\left(\frac{s}{a}\right) + B\right)$$

$$\phi(\infty) = 0 \Rightarrow B = 0 \Rightarrow \phi(s) = A\left(\operatorname{arctg}\left(\frac{s}{a}\right) - \frac{\pi}{2}\right)$$

$$\operatorname{arctg} x = \int \frac{1}{x^2+1}\,dx = \int \frac{1}{x^2}\frac{dx}{1+\frac{1}{x^2}} \approx \int \frac{1}{x^2}\,dx = -\frac{1}{x} + \frac{\pi}{2}$$

$$\Rightarrow \operatorname{arctg}(x) = \frac{\pi}{2} - \frac{1}{x} \quad \text{pro} \quad x \to \infty$$

$$\phi(s) = A\left(\operatorname{arctg}\frac{s}{a} - \frac{\pi}{2}\right) \approx A\left(\frac{\pi}{2} - \frac{a}{s} - \frac{\pi}{2}\right) = -\frac{Aa}{s} = \frac{Q}{4\pi\varepsilon_0 a}$$

$$\Rightarrow \phi(s) = \frac{Q}{4\pi\varepsilon_0 a}\left(\frac{\pi}{2} - \operatorname{arctg}\frac{s}{a}\right)$$

Kapacita vodivého tělesa

$$\phi(T) - \phi(\infty) = U \quad \leftarrow \text{těleso}$$

$$Q = CU$$

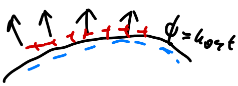
Vodivá koule

$$\phi(r=a) = U = \frac{Q}{4\pi\varepsilon_0 a} \Rightarrow C_{koule} = 4\pi\varepsilon_0 a$$

Vodivý disk

$$\phi(s=0) = \frac{Q}{4\pi\varepsilon_0 a}\,\frac{\pi}{2} = \frac{Q}{8\varepsilon_0 a} \Rightarrow C_{disk} = 8\varepsilon_0 a$$

Důsledek: známe σ na povrchu vodiče

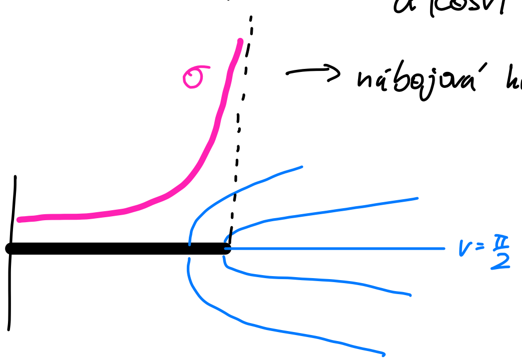


$$\sigma = \varepsilon_0 E_\perp$$

$$E_\perp = \frac{1}{h_s}\,\frac{A}{s^2+a^2} = \sqrt{\frac{s^2+a^2}{s^2+a^2\cos^2 v}}\,\frac{A}{s^2+a^2}$$

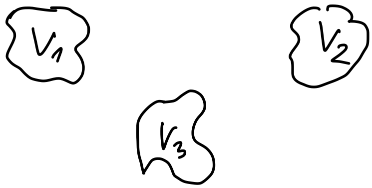
$$E_\perp(s=0) = \frac{A}{a^2|\cos v|}$$

$\Rightarrow$



$\rightarrow$ nábojová hustota jde do ∞

Problém v elektrostatice:



$\phi(\partial V_a) = U_a$ — okraj. podm.

$\nabla^2 \phi = 0$ — polní rovnice

Greenovy věty

$$\nabla\cdot(f\vec{A}) = \nabla f\cdot\vec{A} + f\nabla\cdot\vec{A}$$

pro $\vec{A} = \nabla g$:

$$\nabla\cdot(f\nabla g) = \nabla f\cdot\nabla g + f\nabla^2 g \quad \Big/ \int_\Omega \dots dV$$

$$\oint_{\partial\Omega} f\nabla g = \int \nabla f \nabla g \, dV + \int f\nabla^2 g \, dV$$

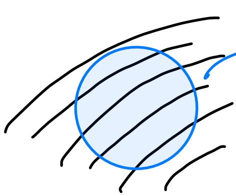
1. Greenova věta

$$\int_\Omega f\nabla^2 g \, d^3x = \oint_{\partial\Omega} f\nabla g \, d\vec{S} - \int_\Omega \nabla f \nabla g \, d^3x$$

2. Greenova věta

$$\int_\Omega (f\nabla^2 g - g\nabla^2 f)\, d^3x = \oint_{\partial\Omega} (f\nabla g - g\nabla f)\, d\vec{S}$$

$\underline{\nabla f = 0}$



extrém f vzhľadom

můžeme:

$g = \frac{1}{r} \qquad \nabla f = 0$

$\Omega = K_a(0)$

PS: $\int_K f \Delta \frac{1}{r} - \frac{1}{r} \cancel{\Delta f}^{\,0} \, d^3x = \int_K f(-4\pi)\delta^3(x)\, d^3x = -4\pi f(0)$

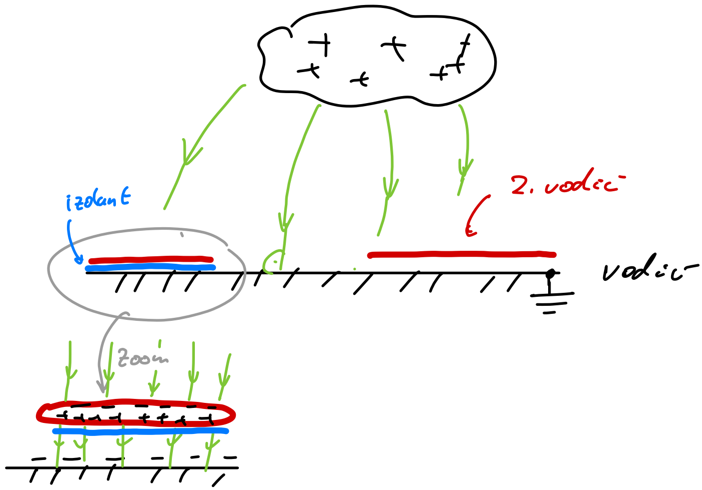
LS: $\oint_{\partial K} \left[f\left(\frac{-\vec{e}_r}{r^2}\right) - \frac{1}{r} \nabla f \right] \cdot d\vec{S} = \frac{1}{-a^2} \oint_{\partial K} f \, d\vec{S}$

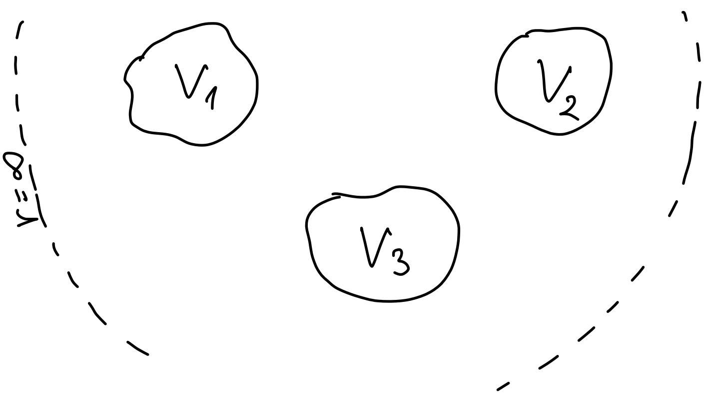
$\frac{1}{r}$ je konst na hran. kole

$-\oint_{\partial K} \nabla \cdot f dS = -\int_{\partial K} \nabla^2 f \, d^3x = 0$

$\Rightarrow f(0) = \int f(\vec{x}) \frac{dS}{4\pi a^2} = \langle f \rangle_{\partial K}$

Na vodiči $\vec{E}_{\parallel} = 0$





$\phi(\vec{x})$; $\vec{x} \in \Omega$

$\Omega = \mathbb{R}^3 \setminus \left(\bigcup_{a=1}^{N} V_a \right)$

$\Delta \phi = 0$

$\phi(\partial V_a) = U_a \quad \phi(r=\infty) = c$

- Nejde:

$$\phi(0)=1, \quad \phi(\infty)=0$$

$$\nabla^2\phi=0 \qquad V_1=\{0\}$$

↳ podobně: V_1 = úsečka, přímka

ϕ_1, ϕ_2 jsou řešení $\Rightarrow \phi_1=\phi_2$

$$16V(\phi_1-\phi_2; \phi_1-\phi_2) = \int_\Omega (\phi_1-\phi_2)\underbrace{\Delta(\phi_1-\phi_2)}_{0}\, d^3x = \oint_{\partial\Omega} \overset{0 \text{ na } \partial\Omega}{\cancel{(\phi_1-\phi_2)}}\nabla(\phi_1-\phi_2)\, d\vec{S} - \int_\Omega [\nabla\phi_1-\phi_2]^2 d^3x$$

$$\Rightarrow \phi_1=\phi_2$$

Kapacita

a) 1. vodič: $Q = CU$

b) více vodičů: $V_1 .. V_N$: vodiče

$U_1 \dots U_N$: $\phi(\partial V_a)=U_a, \quad \phi(\infty)=0$

$$\phi(\vec{x}) = \sum_{a=1}^{N} U_a \, \psi_a(\vec{x})$$

$$\psi_a(\vec{x}) = \begin{cases} 0 & \vec{x}\in\partial V_b \quad b\neq a \\ 1 & \vec{x}\in\partial V_a \end{cases}$$

$$\Delta\psi_a = 0 \qquad \psi(\infty)=0$$

Co náboje:

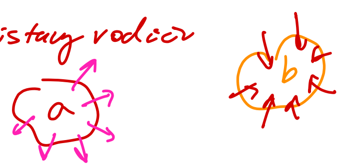
$$Q_a = \varepsilon_0 \int_{\partial V_a} -\nabla\phi \, d\vec{S} = \sum_{b=1}^{N} U_b \, \varepsilon_0 \oint_{\partial V_a} -\nabla\psi_b \, d\vec{S}$$

$$\nabla^2 u = 0$$

$$Q_a = \sum_{b=1}^{N} C_{ab} U_b$$

↳ matice kapacit systému vodičů

$$C_{ab} := \varepsilon_0 \oint_{\partial V_a} -\nabla\psi_b \, d\vec{S}$$



$$\boxed{\partial\Omega = \bigcup_{a=1}^{N}(-\partial V_a) + \partial K_\infty}$$

Ψ_b je 0 mimo ∂V_b
1 v ∂V_b

$$\frac{1}{\varepsilon_0}(C_{ab}-C_{ba}) = \oint_{\partial V_b}\nabla\Psi_a\cdot d\vec S - \oint_{\partial V_a}\nabla\Psi_b\cdot d\vec S = -\oint_{\partial\Omega}\Psi_b\nabla\Psi_a\, d\vec S + \oint_{\partial\Omega}\Psi_a\nabla\Psi_b\cdot d\vec S$$

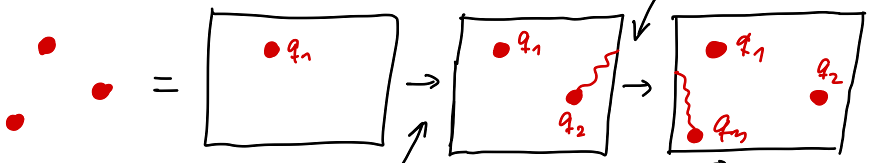
$$= -\int_\Omega \Psi_b\Delta\Psi_a\, d^3x + \int_\Omega \Psi_a\Delta\Psi_b\, d^3x = 0 \Rightarrow C_{ab}=C_{ba}$$

$$C_{ab} = \begin{pmatrix} C_{11} & & \\ & C_{22} & \\ & & \ddots \end{pmatrix} \qquad C_{ab}\begin{cases} \geq 0 & a=b \\ \leq 0 & a\neq b \end{cases}$$

Energia elstat. pole

1) inspirace: sada bodových nábojů

$$\int \vec F\cdot d\vec l = \int q_3(\vec E_1+\vec E_2)\,d\vec l = \dots = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1q_2}{|\vec r_1-\vec r_2|}+\frac{q_1q_2}{|\vec r_2-\vec r_3|}\right.$$



$$W = \int \vec F\cdot d\vec l = \int q_2\,\vec E_1\cdot d\vec l = \int_{r=\infty}^{\vec r_2} q_2(-\nabla\phi_1)\,d\vec l = \frac{q_1q_2}{|\vec r_1-\vec r_2|}\frac{1}{4\pi\varepsilon_0}$$

$$\Rightarrow W = \frac{1}{4\pi\varepsilon_0}\sum_a\sum_{b<a}\frac{q_a q_b}{|\vec r_a-\vec r_b|} = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_a\sum_{b\neq a}\frac{q_a q_b}{|\vec r_a-\vec r_b|} =$$

$$= \frac{1}{2}\sum_a q_a\,\phi^*(\vec x_a) \qquad \phi^*(\vec x_a) = \frac{1}{4\pi\varepsilon_0}\sum_{b=a}\frac{q_b}{|\vec r_a-\vec r_b|}$$

→ ze spojitosti: $W=\frac{1}{2}\int \rho\, \phi_\rho(\vec{x})\, d^3\vec{x}$

$$\rho = -\varepsilon_0 \Delta \phi_\rho$$

$W = -\frac{\varepsilon_0}{2}\int \phi \Delta \phi \, d^3\vec{x} =$ pro spojité ρ (ne distribuci)

pokud $\phi(\infty)=0$, $|\int \nabla\phi| < \infty$

$$\overset{(GV)}{=} -\frac{\varepsilon_0}{2}\oint_{\partial\mathbb{R}^3} \phi \nabla\phi \, d\vec{S} + \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} |\vec{E}|^2 d^3x =$$

$d\vec{S} = \vec{n}\cdot r^2 d\Omega$

$$= \frac{1}{2}\int \varepsilon_0 \vec{E}\cdot\vec{E}\, d^3x$$

$$\Rightarrow \boxed{W = \frac{1}{2}\int \rho\, \phi(\vec{x})\, d^3x = \frac{1}{2}\int \varepsilon_0 \vec{E}\cdot\vec{E}\, d^3x}$$

$$W = \frac{1}{2}\int \sum_i \sigma_a \phi_a \, dS_a = \frac{1}{2}\sum_i Q_a V_a = \frac{1}{2}\sum_{a,b=1}^{N} C_{ab} V_a V_b$$

Pole nábojů za přítomnosti vodičů

ρ dáno... $\phi, \vec{E} = ?$

Komplikace: vodiče

- bez komplikace: $\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\vec{x}-\vec{x}'|}\rho(\vec{x}')\, d^3x'$
- s vodiči: $\phi = \frac{1}{4\pi\varepsilon_0}\int G(\vec{x},\vec{x}')\rho(\vec{x}')\, d^3x'$

$$G_0(\vec{x},\vec{x}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\vec{x}-\vec{x}'|}$$ $\longrightarrow$ potenciál jednotkového bodového náboje v bode $\vec{x}'$ meraný v $\vec{x}$

$$\Delta \frac{1}{r} = -4\pi\,\delta^3$$

$$\Delta \frac{1}{4\pi\varepsilon_0}\frac{1}{|r-r'|} = -\delta^3(\vec{x}-\vec{x}')$$

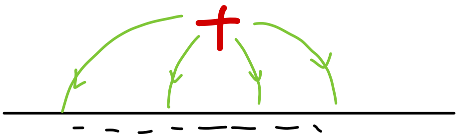
$G(\vec{x},\vec{x}')$ je riešení Poissonovy rovnice v promenné $\vec{x}$ pro zadaný $\varrho = \delta(\vec{x}-\vec{x}')$

$$\Delta G(\vec{x},\vec{x}') = -\frac{1}{\varepsilon_0}\delta^3(\vec{x}-\vec{x}')$$

$\Rightarrow$ rozdíl medzi G_0, G jsou hraničný podmínky

- hraniční podm.: $\phi(\partial\Omega_a) = 0$ (jinak pridáme $\sum V_a \Psi_a(\vec{x})$)

Pr: vodivá rovina $z=0$ a dva bodové náboje



$\phi=0$

$$G(\vec{x},\vec{x}') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\left((x-x')^2+(y-y')^2+(z-z')^2\right)^{\frac{1}{2}}} + \frac{1}{\left((x-x')^2+(y-y')^2+(z+z')^2\right)^{\frac{1}{2}}}\right] =$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{x}-\vec{x}'|} - \frac{1}{|\vec{x}-\vec{x}''|}\right]$$ $\rightarrow$ fiktívny záporný náboj

$\vec{x}'' = [x', y', -z']$

$$\boxed{\sigma = \varepsilon_0 \vec{n}\cdot\vec{E}}$$

$$\vec{E} = -\nabla\vec{E} = -\nabla_x G(\vec{x},\vec{x}')$$

$$\int \sigma\, dS = -q$$

$$\Delta\phi = 0$$

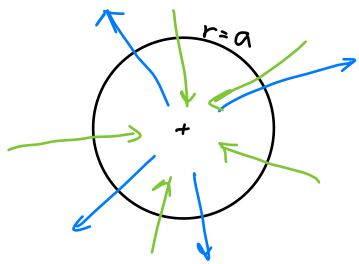
$\phi(x,y,z) \rightarrow \phi(x,y,-z)$ — nezmení Lap. rovnicu

$\rightarrow$ pre $\Delta\phi = \chi \rightarrow \phi' = \phi(+z) - \phi(-z)$

Kulová inverze

$$\phi(r,\theta,\varphi)$$

$$\Delta\phi(r,\theta,\varphi) = \chi(r,\theta,\varphi)$$



$$\phi(r,\theta,\varphi) \rightarrow \phi = \left(\frac{a^2}{r}, \theta, \varphi\right)$$

$$\Delta\left[\frac{a}{r}\phi\left(\frac{a^2}{r}, \theta, \varphi\right)\right] = \frac{a^5}{r^5}\chi\left(\frac{a^2}{r}, \theta, \varphi\right)$$

$$\phi = 1 \longmapsto \Delta\phi = 0$$

$$\phi = \frac{a}{r} r$$

$$\phi = z \quad (\vec{E} = -\vec{e}_z)$$

$$\phi = \frac{a}{r}\left(\frac{a^2}{r}\right)\cos\theta = \frac{a^3}{r^2}\cos\theta = \frac{a^3}{r^3} z$$

$\rightarrow$ dipól !

$$\phi_{dipól} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

$$\phi(\vec{r}) = \frac{1}{\varepsilon_0}\int \frac{\rho(\vec{r}')}{4\pi|r-r'|} d^3r'$$

$$\phi = \frac{1}{\varepsilon_0}\int G(\vec{r},\vec{r}')\rho(\vec{r}') d^3r'$$

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi}\left[\frac{1}{|\vec{r}-\vec{r}'|} - \frac{a}{|\vec{r}'|}\frac{1}{|\vec{r}-\vec{r}''|}\right] \qquad \vec{r}'' = \frac{a^2}{r^2}\vec{r}'$$

Řešení Laplaca ($\Delta\phi = 0$) ve sférických souřadnicích

$$\Delta\phi = \frac{1}{h_r h_\theta h_\varphi}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_\varphi}{h_r}\frac{\partial}{\partial r}\phi\right) + \frac{\partial}{\partial\theta}\left(\frac{h_r h_\varphi}{h_\theta}\frac{\partial}{\partial\theta}\phi\right) + \frac{\partial}{\partial\varphi}\left(\frac{h_r h_\theta}{h_\varphi}\frac{\partial}{\partial\varphi}\phi\right)\right) =$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\phi\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = f(r)g(\theta)h(\varphi)$$

$$\frac{\Delta\phi}{\phi} = \frac{1}{rf}(rf)'' + \frac{1}{r^2}\frac{1}{g}\frac{1}{\sin\theta}(\sin\theta\, g')' + \frac{1}{r^2}\frac{h''}{h\sin^2\theta} = 0$$

vnitřní řešení

$\to$ obecně: $f = Dr^\ell + \frac{E}{r^{\ell+1}}$

vnější řeš.

$\Rightarrow \quad r\frac{(rf)''}{f} = \text{konst} \qquad$ Ansatz: $f = r^\ell$: $\quad \frac{r(r\,r^\ell)''}{r^\ell} = (\ell+1)\ell = \text{konst}$

$\Rightarrow \quad \frac{h''}{h} = \text{konst} \quad \longrightarrow \quad h'' + (-\text{konst})h = 0 \quad \Rightarrow \quad h = e^{im\varphi} \qquad m \in \mathbb{Z}$

$\to$ obecně: $h = A\cos m\varphi + B\sin m\varphi$

$g = ?$ $\to$ obecně závisí od m, ℓ : $g(\theta) = P_\ell^m(\cos\theta)$

splňují $\Delta\phi$:

$$\left\{\begin{array}{llll} 1 & & & \\ x & y & z & \\ xy & xz & yz & ,\ \underbrace{x^2-y^2,\ y^2-z^2,\ z^2-x^2}_{\text{ale } \sum = 0} \end{array}\right.$$

$$e^{im\varphi} = \cos m\varphi + i\sin m\varphi = (\cos\varphi + i\sin\varphi)^m = \left(\frac{x+iy}{r\sin\theta}\right)^m$$

$$P(x,y,z) \longrightarrow P(x+iy, z, r)$$

polynomy 2 řádu

$r^2 - 3z^2$, $z(x+iy)$, $z(x-iy)$, $(x+iy)^2$, $(x-iy)^2$

polynomy 3 stupně

$r^3 - 5rz^2$, $(r^2 - A'z^2)(x+iy)$; $(r^2 - A'z^2)(x-iy)$, $z(x+iy)^2$, $z(x-iy)^2$; $(x+iy)^3$

$$\Delta\left(r^{\ell} P_{\ell}^{m}(\cos\theta) e^{im\varphi}\right) = 0$$

$$r^2 \cos\theta \left(\sin\theta\cos\varphi + i \sin\theta \sin\varphi\right) =$$
$$r^2 (\cos\theta \sin\theta)(\cos\varphi + i\sin\varphi)$$

$\ell = 2$ $\qquad m = 1 \qquad \Rightarrow P_2^1 \sim \cos\theta\sin\theta$

Separované řešení $\qquad \Delta\phi_{sep} = 0$

$$\phi_{sep} = \left(Ar^{\ell} + \frac{B}{r^{\ell+1}}\right) Y_{\ell m}(\theta, \varphi)$$

$Y_{\ell m} = \text{konst } P_{\ell}^{m}(\cos\theta) e^{im\varphi}$ $\quad \to$ báze funkcí na kouli

$\ell = 0, 1, \dots \quad m = -\ell, \dots, \ell$

$u(\theta, \varphi) \leftrightarrow U_{\ell m}$

$$u(\theta, \varphi) = \sum_{\ell, m} U_{\ell m} Y_{\ell m}(\theta, \varphi)$$

$\to d\Omega = \sin\theta \, d\theta d\varphi$

$$U_{\ell m} = \int u(\theta, \varphi) Y_{\ell m}^{*} d\Omega = \int \left(\sum_{\ell, m} U_{\ell' m'} Y_{\ell' m'}\right) Y_{\ell m}^{*} d\Omega =$$

$$= \sum_{\ell, m} U_{\ell' m'} \int Y_{\ell' m'} Y_{\ell m}^{*} d\Omega \Rightarrow \int Y_{\ell' m'} Y_{\ell m}^{*} d\Omega = \delta_{\ell\ell'}\delta_{mm'}$$

$$\to Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$Y_{\ell m} \to$ báze fcí na sféře

$$\phi = \dots \to \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

$$Q = \oint -\nabla\phi \, d\vec{S} = \int \rho \, d^3x$$

$$\int_\Omega \Delta\phi \, d^3x = \oint \nabla\phi \cdot d\vec{S}$$

(vlastn. zdroje) ~ (pole daleko)

$$\int_\Omega (\psi\Delta\phi - \phi\Delta\psi)\, d^3x = \oint_{\partial\Omega} (\psi\nabla\phi - \phi\nabla\psi)\cdot d\vec{S}$$

$$\Omega = K_r(0)$$

$$\psi = r^{l'} Y^*_{l'm'}$$

$$\phi = \sum_{\substack{l=0,1,2..\\ m=-l,..l}} \phi_{lm} \frac{Y_{lm}(r,\varphi)}{r^{l+1}}$$

$$\nabla\phi\cdot d\vec{S} = \nabla\phi\cdot\vec{e}_r\, dS = \partial_r\phi\, dS = \sum_{l,m} -(l+1)\phi_{lm}\frac{Y_{lm}}{r^{l+2}}\, dS$$

$$\nabla\psi\cdot d\vec{S} = \partial_r\psi\, dS = l' r^{l'-1} Y^*_{l'm'}\, dS$$

$$\oint\psi\nabla\phi\cdot d\vec{S} = \int_K r^{l'}Y^*_{l'm'}\sum_{lm} -(l+1)\phi_{lm}\frac{Y_{lm}}{r^{l+2}}\, r^2 d\Omega =$$

$$= \sum_{lm}\frac{r^{l'}}{r^{l+2}}(-l+1)\phi_{lm}\int_K \underbrace{Y^*_{l'm'}Y_{lm}}_{\delta_{l'l}\delta_{mm'}} r^2 d\Omega = \phi_{l'm'}(-l'+1)$$

$$\oint\phi\nabla\psi = \int\sum \phi_{lm}\frac{Y_{lm}}{r^{l+1}}\, l' r^{l'-1} Y^*_{l'm'} r^2 d\Omega = \dots = \phi_{l'm'}\, l'$$

$$\Rightarrow -\frac{1}{\varepsilon_0}\int \rho(\vec{x})\, r^l\, Y^*_{l'm'} = -(2l+1)\phi_{l'm'}$$

$$\Delta \phi(r,\theta,\varphi) = 0 \qquad \phi \sim \left(A r^{\ell} + \frac{B}{r^{\ell+1}}\right) \underbrace{P_\ell^m(\cos\theta) e^{im\varphi}}_{Y_{\ell m}(\theta,\varphi)}$$

$$\| Y_{\ell m} \| = 1$$

$$\int Y_{\ell m}(\theta,\varphi) Y_{\ell' m'}(\theta,\varphi)\, d\Omega = \delta_{\ell\ell'} \delta_{mm'}$$

TVRZ 2: vně zdroje uvnitř koule $K_a(0)$.

$$\phi = \sum_{\substack{\ell \in \mathbb{N}_0 \\ |m| < \ell}} \phi_{\ell m} \frac{Y_{\ell m}(\theta,\varphi)}{r^{\ell+1}}$$

$\to$ splňuje h.p. a $\Delta \phi = 0 \implies$ Řešení!!!

II GV: $\left(\int \text{přes zdroje}\right) = \left(\int \text{přes dalekou sféru}\right)$

$$-\frac{1}{\varepsilon_0} \int \varrho r^{\ell} Y^*_{\ell' m'} = -(2\ell'+1)\, \phi_{\ell' m'}$$

$$\varrho(r,\varphi,\theta) \longrightarrow \phi_{00}, \phi_{1-1}, \dots$$

$$r^{\ell} P_\ell^m(\cos\theta) e^{im\varphi}$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\varrho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d^3 r'$$

$$f(x',y',z') = f(0,0,0) + \left.\frac{\partial f}{\partial x'}\right|_0 x' + \left.\frac{\partial f}{\partial y'}\right|_0 y' + \left.\frac{\partial f}{\partial z'}\right|_0 z'$$
$$+ \frac{1}{2}\left(\left.\frac{\partial^2 f}{\partial x'^2}\right|_0 x' + \left.\frac{\partial^2 f}{\partial y'^2}\right|_0 y' + \left.\frac{\partial^2 f}{\partial z'^2}\right|_0 z' + \dots \right) + \dots$$

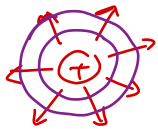
$$\frac{1}{|\vec r-\vec r'|} = \frac{1}{|\vec r|} + \underbrace{\left(\partial_i' \frac{1}{|\vec r-\vec r'|}\right)\Big|_{r'=0}}_{\frac{r_i}{r^3}} r_i' + \frac{1}{2!}\underbrace{\left(\partial_{ij}' \frac{1}{|\vec r-\vec r'|}\right)\Big|_{r'=0}}_{\frac{3r_ir_j - r^2\delta_{ij}}{r^5}} r_i'r_j' + \frac{1}{3!}\underbrace{\left(\partial_{ijk}' \frac{1}{|\vec r-\vec r'|}\right)\Big|_{r'=0}}_{\frac{15 r_ir_jr_k - 9\delta_{(ij}r_{k)}r^2}{r^7}} r_i'r_j'r_k'$$

$$4\pi\varepsilon_0\phi(\vec r) = \overbrace{\frac{1}{r}\int \rho' d^3r'}^{Q} + \overbrace{\frac{r_i}{r^3}\int r_i'\rho' d^3x}^{p_i - \text{dipól}} + \overbrace{\frac{3r_ir_j-\delta_{ij}r^2}{r^5}\int r_i'r_j'\rho' d^3x'}^{\text{kvadrupól}} + \dots$$

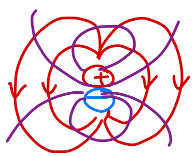
$$\phi(\vec r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\vec p\cdot\vec r}{r^3} + \frac{3r_ir_j - r^2\delta_{ij}}{r^5}Q_{ij} + \dots\right]$$

$$\frac{1}{|\vec r-\vec r'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$

- nábojový člen $\frac{Q}{r}$

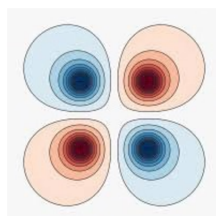


- dipólový člen $\frac{\vec p\cdot\vec r}{r^3}$



- quadrupólový člen $\frac{3r_ir_j - r^2\delta_{ij}}{r^5}Q_{ij}$

$\quad\quad Q_{ij} \to$ 5 nezávislých zložiek



- octupólový člen

$\vdots$

→ rozvoj konverguje pre $r > r_0$, kde r_0 je koule, ktorá obklopuje všetky náboje

→ axiálne symetrický zdroj:

$$\Phi_{lm} = \int \rho\, r^l \overset{e^{im\varphi}}{Y^*_{lm}}(\theta,\varphi)\, d^3x = \int \rho\, r^l P^l_0(\cos\theta)\, d^3x$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} q_l \frac{P_l(\cos\theta)}{r^{l+1}}$$

$$q_l = \int \rho(r')\, r'^{l'} P_{l'}(\cos\theta')\, d^3x'$$

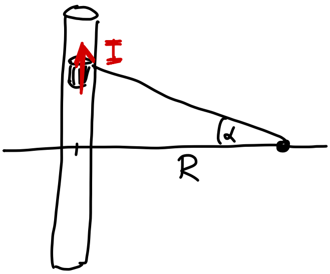
→ nabitá tyčka:

$$\cos\theta = \begin{cases} +1 \\ -1 \end{cases} = \mathrm{sgn}(z)$$

$$r^{l'} P_l(\cos\theta) = |z|^l P_l(\mathrm{sgn}\, z) = z^l$$

$$\Rightarrow q_l = \int_{-a}^{a} \frac{Q}{2a} z^l dz = \begin{cases} q_0, q_2, q_4, \dots \neq 0 \\ q_1, q_3, q_5, \dots = 0 \end{cases}$$

Magnetizmus



$$z' = R\,tg\,\alpha \qquad dQ' = \lambda\, dz'$$

$$\vec{x}\,' = [0, 0, R\,tg\,\alpha] \qquad dz' = \frac{R}{\cos^2\alpha}\, d\alpha$$

$$\vec{x} = [R, 0, 0]$$

$$|\vec{x} - \vec{x}\,'|^2 = R^2 + R^2 tg^2 = \frac{R^2}{\cos^2\alpha}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int dQ' \, \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} = \frac{1}{4\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\lambda R}{\cos^2\alpha}\, d\alpha \, \frac{[R, 0, -R\,tg\,\alpha]}{(R/\cos\alpha)^3}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha \, d\alpha \, \vec{e}_x = \frac{\lambda}{2\pi\varepsilon_0 R}\, \vec{e}_x$$

$$dQ' \, \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3} \;\underset{\text{elstat.}}{\overset{\text{magstat.}}{\rightleftarrows}}\; d\vec{J}\,' \times \frac{\vec{x} - \vec{x}\,'}{|\vec{x} - \vec{x}\,'|^3}$$

$$dJ = \begin{cases} \vec{j}\, d^3x \\ \vec{j}_{plocha}\, dS \\ I\, d\vec{\ell} \end{cases} \qquad [j] = \frac{A}{m^2}, \quad [j_{ploch}] = \frac{A}{m}, \quad [I] = I$$

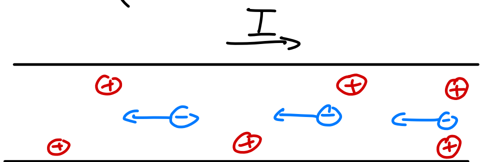
$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

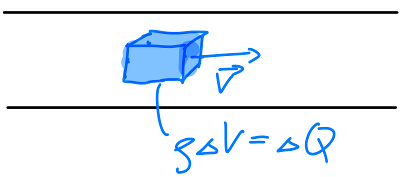
$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} F^{\alpha\beta} \qquad \frac{\partial x'}{\partial x} = \Lambda = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \gamma = 1 + O\left(\frac{v^2}{c^2}\right)$$

$$\boxed{\begin{array}{ll} E'_\parallel = E_\parallel & \vec{E}'_\perp = \gamma\left(\vec{E}_\perp + \frac{\vec{v}}{c}\times c\vec{B}\right) \\ B'_\parallel = B_\parallel & c\vec{B}'_\perp = \gamma\left(c\vec{B}_\perp - \frac{\vec{v}}{c}\times \vec{E}_\perp\right) \end{array}}$$

→ pro malé rychlosti (pomalá Lorentzova trans.)

$$\Lambda = \begin{pmatrix} 1 & \frac{\vec{v}}{c} \\ -\frac{\vec{v}}{c} & 1 \end{pmatrix} \Rightarrow \vec{B}' = \vec{B} - \frac{\vec{v}}{c^2}\times\vec{E}$$

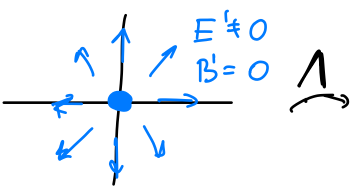


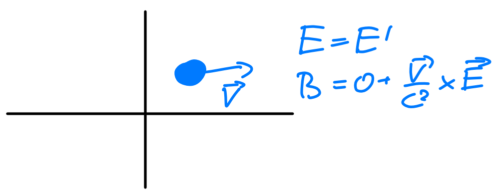


$$v\frac{\Delta Q}{\Delta S} = I\,\Delta t_v$$

$\vec{j} = \rho\vec{v}$ (1 komponent)

$$\vec{B} = \vec{B}' + \frac{\vec{v}}{c^2}\times\vec{E}$$





electron frame vodič frame

$dQ'\vec{v} = d^3x'\rho\vec{v} = \vec{j}\,d^3x'$

$$dB = \frac{\vec{v}}{c^2}\times d\vec{E} = \frac{\vec{v}}{c^2}\times\left(\frac{1}{4\pi\varepsilon_0}\,dQ'\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) = \frac{\mu_0}{4\pi}\,d^3x'\,\vec{j}\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$\Rightarrow$ Biot-Savart je dôsledok relativity

Stacionární proudy a pole

$j, B \neq 0$, ale $\frac{\partial}{\partial t}$ cokoliv $= 0$

$\uparrow E, \rho, B, \vec{j}, \phi$

$$\boxed{\begin{aligned} \nabla \times \vec{B} &= \mu_0 \vec{j} \\ \nabla \cdot \vec{B} &= 0 \end{aligned}}$$

→ nezávisí na $\vec{E}$ → magnetostatika je nezávislá od $\vec{E}$

$\exists \vec{A}: \quad \vec{B} = \nabla \times \vec{A}$ (vek. potenciál) vždy, lebo $\nabla \cdot \nabla \times \vec{A} = 0$

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$\vec{A}' = \vec{A} + \nabla \chi \quad \Rightarrow \quad \nabla \times \vec{A}' = \nabla \times \vec{A}$, lebo $\nabla \times \nabla \chi = 0$

$\vec{A}$ lze změnit, aniž změním $\vec{B}$ přidám ∇ libovolné funkce

$$\oint_{\partial S} \vec{A} \cdot d\vec{l} = \int_S \nabla \times \vec{A} \cdot d\vec{S} = \int_S \vec{B} \cdot d\vec{S} \ \dots \text{ mag. tok}$$

$\Rightarrow$ $\boxed{\nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \chi}$, pak nalezením $\chi: \nabla^2 \chi = -\nabla \cdot \vec{A}$ dostaneme $\nabla \cdot \vec{A}' = 0$

$\uparrow$ nenulové $\nabla \cdot \vec{A}$

Amperov zákon pro potenciál:

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{j}}$$

analogie:

$$\nabla^2 \phi = -\frac{1}{\varepsilon_0} \rho \quad \rightarrow \quad \phi = \frac{1}{4\pi} \cdot \frac{1}{\varepsilon_0} \int \frac{\rho'}{|\vec{x} - \vec{x}'|} d^3x'$$

$$\vec{A} = \frac{1}{4\pi} \mu_0 \int \frac{\vec{j}'}{|\vec{x} - \vec{x}'|} d^3x'$$

$$\boxed{\vec{A} = \frac{1}{4\pi}\mu_0 \int \frac{\vec{j}'}{|\vec{x}-\vec{x}'|} d^3x'}$$

→ platí len v Descartezu

$$\left[\nabla \frac{1}{|\vec{x}-\vec{x}'|} = -\frac{1}{|\vec{x}-\vec{x}'|^2} \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}\right]$$

$$\vec{B} = \nabla\times\vec{A} = \nabla\times\int \frac{\mu_0}{4\pi}\frac{\vec{j}(x')}{|\vec{x}-\vec{x}'|}d^3x'$$

$\vec{A}$ je konst v $\vec{x}$

$$\nabla\times(f\vec{A}) = \nabla f\times\vec{A} + \cancel{f\nabla\times\vec{A}} = \nabla f\times\vec{A}$$

$$= \left| f = \frac{1}{\vec{x}-\vec{x}'}\right| = -\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\times\vec{A}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int \vec{j}(\vec{x}')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3x'}$$

$\vec{j}(\vec{x}')$ je konst. v $\vec{x}$

$$\nabla\cdot\vec{A} = \int\frac{\mu_0}{4\pi}\left[\nabla\frac{1}{|\vec{x}-\vec{x}'|}\cdot\vec{j}' + \cancel{\frac{1}{|\vec{x}-\vec{x}'|}\nabla\cdot\vec{j}'}\right]d^3x' =$$

$$= \int\frac{\mu_0}{4\pi} - \nabla'\frac{1}{|\vec{x}-\vec{x}'|}\cdot\vec{j}' = \int\frac{\mu_0}{4\pi}\left[\nabla'\cdot\left[\frac{\vec{j}'}{|\vec{x}-\vec{x}'|}\right] - f\underbrace{\nabla'\cdot\vec{j}'}_{0}\right]d^3x =$$

$$\left[\Delta\vec{A} = -\mu_0\vec{j} \rightarrow \Delta(\nabla\cdot\vec{A}) = -\mu_0(\nabla\cdot\vec{j})\right.$$

$\Rightarrow \nabla\cdot\vec{j} = 0$ je podm. správnosti rovnice $\Delta\vec{A} = -\mu_0\vec{j}$ a je splnená, pretože $\partial_t\varrho = 0$ pro stac. pole.]

$$= \oint_{\partial S}\frac{\mu_0}{4\pi}\frac{\vec{j}}{|\vec{x}-\vec{x}'|}\cdot d\vec{S} = 0$$

vyžadujeme $\vec{j}(\infty) = 0$

$$\nabla\times\vec{B} = -\mu_0\vec{j} \quad /\nabla\times$$

$$\Rightarrow \nabla(\cancelto{0}{\nabla\cdot\vec{B}}) - \Delta\vec{B} = \mu_0\nabla\times\vec{j} \Rightarrow \Delta\vec{B} = -\mu_0\nabla\times\vec{j}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\vec{j}'}{|\vec{x}-\vec{x}'|}\,d^3x'}$$

Integrální verze Ampéra

$$\nabla\times\vec{B} = \mu_0\vec{j} \quad \text{... diff. verze}$$

Integ. verze Ampera

$$\int_S \nabla\times\vec{B}\cdot d\vec{S} \overset{\text{Stokes}}{=} \boxed{\oint_{\partial S}\vec{B}\cdot d\vec{l} = \int\mu_0\vec{j}\cdot d\vec{S} = \mu_0 I}$$

stačí axiální symetrie:

$$2\pi R B_\varphi = \mu_0 I$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla\times\vec{E} = 0$$
$$\vec{E} = -\nabla\phi$$
$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$
$$\phi(x) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$
$$\vec{E}(x) = \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\rho(x')\,\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$\nabla\times\vec{B} = \mu_0\vec{j} \quad \overset{!}{\Rightarrow} \quad 0 = \nabla\cdot\vec{j}$$
$$\nabla\cdot\vec{B} = 0$$
$$\vec{B} = \nabla\times\vec{A}$$
$$\nabla^2\vec{A} = -\mu_0\vec{j} \quad (\text{při kalibraci } \nabla\cdot\vec{A} = 0)$$
$$\vec{A}(x) = \frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$
$$\vec{B}(x) = \frac{\mu_0}{4\pi}\int dJ'\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|x|} + \frac{\vec{x}\cdot\vec{x}'}{|x|^3} + \cdots$$

$$\Downarrow$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\int \frac{\vec{j}(x')}{|\vec{x}|} d^3x' + \int \frac{\vec{j}(\vec{x}')\,\vec{x}\cdot\vec{x}'}{|x|^3} d^3x' + \cdots\right] =$$

$$I_k = \int j_k(\vec{x}')\, d^3x' \quad ; \quad I_{kl} = \int j_k(\vec{x})\, x_l\, d^3x$$

$$\frac{\partial}{\partial x_k}(j_k x_l) = \underbrace{\frac{\partial j_k}{\partial x_k}}_{\nabla\cdot\vec{j}=0} x_l + j_k \frac{\partial x_l}{\partial x_k} = 0 + j_l$$

$$\int \frac{\partial}{\partial x_l}(j_l x_k)\, d^3x \overset{\text{Gauss}}{=} \oint j_l x_k\, dS_l = \int j_k\, d^3x$$

Ale $\oint j_l x_k\, dS_l = 0$ pro dostatočne veľkú plochu $S \Rightarrow \int j_k\, d^3x = 0$

$$\Rightarrow A_i = \frac{\mu_0}{4\pi}\int \frac{j'_i\, x_j\, x'_j}{|x_j|^3} d^3x' + \ldots = \frac{\mu_0}{4\pi} I_{ij} \frac{x_j}{|x_j|^3} + \ldots$$

$$I_{kl} + I_{lk} = \int j_k x_l + j_l x_k\, d^3x = \int \frac{\partial}{\partial x_m}(j_m x_k x_l) = 0$$

$j_m \delta_{mk} = j_m \frac{\partial x_m}{\partial x_k}$

$$\frac{\partial}{\partial x_m}(j_m x_k x_l) = \cancel{\frac{\partial j_m}{\partial x_m}}^{\,0} \ldots + j_m x_l \delta_{km} + j_m x_k \delta_{lm}$$

$\Rightarrow I_{kl}$ je antisymetrický

$\hookrightarrow$ V 3D existuje $m \in \mathbb{R}^3$: $I_{kl} = -\varepsilon_{klm} m_m$

$$\Rightarrow m_i = -\frac{1}{2}\varepsilon_{ikl} I_{kl} \overset{\text{DK}}{=} \frac{1}{2}\varepsilon_{ikl}\varepsilon_{klm} m_m = \frac{1}{2}(\underbrace{\delta_{ll}}_{3}\delta_{im} - \underbrace{\delta_{lm}\delta_{il}}_{\delta_{mi}}) m_m$$

$$= \frac{1}{2}(3-1)\delta_{im} m_m = m_i$$

$$\Rightarrow m_i = -\frac{1}{2}\int \varepsilon_{ikl}\, j_k x_l\, d^3x \Rightarrow \boxed{\vec{m} = \frac{1}{2}\int \vec{x}\times\vec{j}\, d^3x}$$

$\hookrightarrow$ mag. dipolový moment

$$I_{ij} x_j = -\varepsilon_{ijm} m_m x_j = -\vec{x}\times\vec{m} = \vec{m}\times\vec{x}$$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{m\times\vec{x}}{|\vec{x}|^3}} = \frac{\mu_0}{4\pi}\frac{m\times\vec{r}}{r^3}$$ vek. potenciál mag. dipólu

$$\boxed{\phi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}}$$ skalarní potenciál elek. dipólu

$$\vec{E}_{dip} = \frac{1}{4\pi\varepsilon_0}\frac{3\vec{p}\cdot\vec{r}\,\vec{r} - \vec{p}r^2}{r^5} \qquad \vec{p} = \int \vec{r}\rho\, d^3r$$

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{3\vec{m}\cdot\vec{r}\,\vec{r} - \vec{m}r^2}{r^5} \qquad \vec{m} = \frac{1}{2}\int \vec{r}\times\vec{j}\, d^3r$$

$$\vec{E} = -\nabla\phi$$

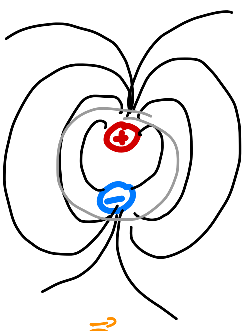
$$\phi = \sum_l\sum_m \phi_{lm}\frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \ ; \ \phi_{lm} = \frac{1}{2l+1}\frac{1}{\varepsilon_0}\int dq\, r^l Y^*_{lm}$$

$\vec{B} = -\nabla\psi \leftarrow$ mag. potenciál (Existuje je vně lok. zdroju)

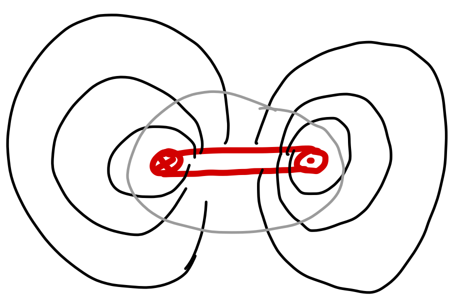
$$\psi = \frac{\mu_0}{4\pi}\sum_l\sum_m \frac{4\pi}{2l+1} M_{lm}\frac{Y_{lm}}{r^{l+1}} \qquad M_{lm} = \frac{1}{l+1}\int d^3x\, r^l Y^*_{lm}\,\vec{r}\cdot(\nabla\times\vec{j})$$

$$M_{10}, M_{1\pm1} \rightarrow m_z,\ m_x \pm i m_y$$

$$\vec{A} = \frac{\mu_0}{4\pi}\sum_l\sum_m \frac{4\pi}{l(l+1)} M_{lm}\left[-\vec{r}\times\nabla\right]\frac{Y_{lm}}{r^{l+1}}$$



$\vec{p}$



$\vec{m}$

$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{p_z\cos\theta}{r^2} \qquad \vec{A} = \frac{\mu_0}{4\pi} m_z\frac{\sin\theta}{r^2}\vec{e}_\varphi$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$dq = \rho\, d^3x \quad \Rightarrow \quad \vec{f} = \rho\vec{E} + \vec{j} \times \vec{B}$$

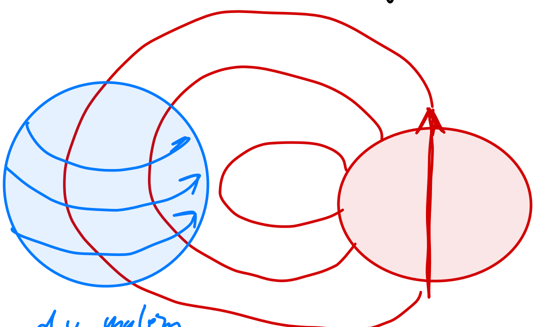
$$\vec{j} = \rho\vec{v}$$

(případně $\vec{j} = \sum_a \vec{v}_a \rho_a$)

$$d\vec{J} = \vec{j}\, d^3x$$

$$\vec{F} = \int \vec{j} \times \vec{B}\, d^3x = \int d\vec{J} \times \int \frac{\mu_0}{4\pi}\, d\vec{J}' \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} =$$

$$= \iint \frac{\mu_0}{4\pi}\, d\vec{J}' \left(d\vec{J} \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \right) - \int (d\vec{J} \cdot d\vec{J}')\, \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}$$



proud v malém objektu

pole vzdáleného zdroje

$$\vec{F} = \int \vec{j} \times \vec{B}\, d^3x$$

1) nechť B je homogenní $\Rightarrow \vec{F}_1 = \vec{B} \times \overbrace{\int \vec{j}\, d^3x}^{0} = 0$

2) $B_i = B_i(x=0) + \left.\dfrac{\partial B_i}{\partial x_j}\right|_{x=0} x_j + \ldots$

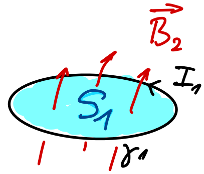
$$F_{2l} = \int \varepsilon_{lki}\, j_k \left(\frac{\partial B_i}{\partial x_j} x_j \right) = \frac{\partial B_i}{\partial x_j} \varepsilon_{jki} \overbrace{\int x_j j_k\, d^3x}^{I_{kj}} =$$

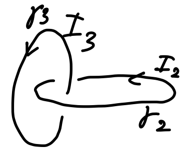
$$= \frac{\partial B_i}{\partial x_j} \varepsilon_{lki} \varepsilon_{jkm} m_m = \frac{\partial B_i}{\partial x_j} (\delta_{lj}\delta_{im} - \cancel{\delta_{lm}\delta_{ij}}^{\nabla\cdot B = 0}) m_m = m_i \frac{\partial B_i}{\partial x_l}$$

$$\Rightarrow \vec{F}_2 = (\vec{m} \cdot \nabla)\vec{B}$$

Magnetické pole smyček

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}'}{|\vec{x}-\vec{x}'|}\, d^3x'$$





$$d\vec{j} = \vec{j}\, d^3x = I\vec{t}\, ds = I\, d\vec{s}$$

$$\Psi_{\text{skrze } \gamma_1 \text{ od } \gamma_2} = \int_{S_1} \vec{B}_2 \cdot d\vec{S}_1 = \int_{S_1} (\nabla \times \vec{A}_2)\cdot d\vec{S}_1 = \int_{\gamma_1} \vec{A}_2 \cdot d\vec{s}_1 =$$

$$= \frac{\mu_0}{4\pi} I_2 \left[\int_{\gamma_1}\int_{\gamma_2} \frac{1}{|\vec{x}_1-\vec{x}_2|}\, d\vec{s}_1 \cdot d\vec{s}_2\right] = L_{12} I_2$$

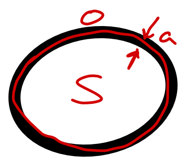
$\Rightarrow$ $\Psi_{\text{skrze } \gamma_1 \text{ od } \gamma_2} = L_{12} I_2$ → 1 príspevok

↳ obecne: $\boxed{L_{kl} = \frac{\mu_0}{4\pi}\int_{\gamma_k}\int_{\gamma_l} \frac{d\vec{s}_k \cdot d\vec{s}_l}{|\vec{x}_k-\vec{x}_l|}}$ → matice indukčnosti

- nedá sa použiť na samoindukčnosť

↳ treba nahradiť:

$$L_{kk} = \frac{\mu_0}{4\pi I^2}\int_{\gamma_k}\int_{\gamma_k} \frac{\vec{j}(\vec{x}_1)\,\vec{j}(\vec{x}_2)}{|\vec{x}_1-\vec{x}_2|}\, d^3x_1\, d^3x_2$$



↳ odhad (spadol z neba): $L \approx \frac{\mu_0}{4\pi}\, o \left(\ln\frac{S}{a^2} + \frac{1}{2} + \varepsilon\right)$

$$\Psi_k = \sum_l \Psi_{\text{skrze } k \text{ od } l} = \sum_l L_{kl} I_l$$

Kvazistacionárne priblíženie

$$\nabla\cdot\vec B = 0 \qquad \nabla\cdot\vec E = \frac{1}{\varepsilon_0}\rho$$
$$\nabla\times\vec B = \mu_0\vec j \qquad \nabla\cdot\vec E = -\frac{\partial\vec B}{\partial t}$$

Faradayov zákon – prom. mag. pole

• konšt. smyčka $\gamma = \partial S$

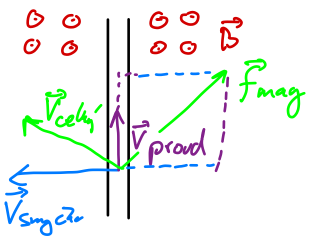
$$-\frac{\partial\vec B}{\partial t} = \nabla\times\vec E \quad \Big/ \int_S \cdot\, d\vec S$$

$$-\frac{d}{dt}\int_S \vec B\cdot d\vec S = \int_S \nabla\times E\, d\vec S = \int_\gamma \overbrace{\vec E}^{\frac{d\vec f_E}{dq}}\circ d\vec s = \mathcal{E}$$

→ elektromotorická sila
↳ spôsobená elek. silou

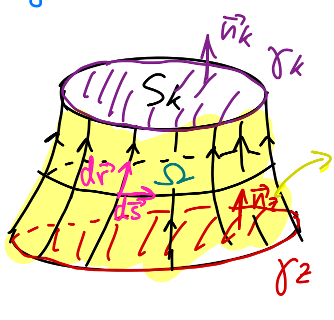
$$\Rightarrow \boxed{-\frac{d\Psi}{dt} = \mathcal{E}}$$ → Faradayov zákon

Faradayov zákon – pohyb smyčky



$\vec V_{smyčka} + \cancel{\vec V_{prúd}}$ → 0, lebo $\parallel \vec B$

$$\mathcal{E} = \int_\gamma \frac{d\vec f_M}{dq}\cdot d\vec s = \int_\gamma \left(\vec V_{celk.}\times\vec B\right)\cdot d\vec s =$$
$$= \int_\gamma \vec V_{smyčka}\times\vec B\cdot d\vec s$$



$$0 = \int_\Omega \nabla\cdot\vec B\, dV = \int_{\partial S}\vec B\cdot d\vec S = \overbrace{\Psi_k - \Psi_z}^{\Delta\Psi} + \int_P \vec B\cdot d\vec S =$$

plášť P → $= \left[\, d\vec S = d\vec s\times \underbrace{d\vec r}_{\text{pohyb smyčky}} = \underbrace{d\vec s}_{\text{podél smyčky}}\times\vec V_s\, dt \,\right] =$

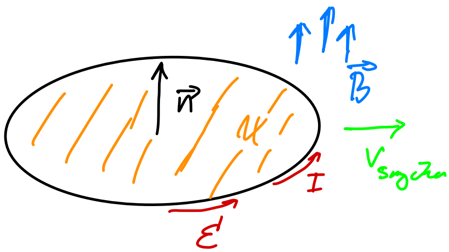
$$= \Delta\Psi + \int_{t_z}^{t_k}\int_\gamma \vec V_s\times\vec B\cdot d\vec s\, dt =$$

$$0 = \Delta\Psi + \int_{t_z}^{t_k}\mathcal{E}\, dt \overset{\text{krátky čas}}{\Rightarrow} -\frac{d\Psi}{dt} = \mathcal{E}$$

premenné $\vec B$ — pohyb

$$\boxed{-\frac{d\Psi}{dt} = \mathcal{E} = \int_\gamma \frac{d\vec f_{EM}}{dt}\cdot d\vec s = \int_\gamma \left(\vec E + \vec v \times \vec B\right)\cdot d\vec s} \rightarrow \text{Faradyov zákon}$$

Lenzovo pravidlo



Práce vykonaná zdroje – proud ve smyčka

← práce

$$\frac{dA_{zdroj}}{dt} = -\frac{dq}{dt}\,\mathcal{E} = I\,\frac{d\Psi}{dt}$$

1) **konst. proud:**

$$\Delta A_{zroj} = I\,\Delta\Psi$$

2) **energie jedné smyčky**

$$\frac{dU_s}{dt} = \frac{dA_{zdroj}}{dt} = I_s\frac{d\Psi_s}{dt} = L_{ss}\,I_s\,\frac{dI_s}{dt}$$

$$\Rightarrow \quad U_s = \frac{1}{2}L_{ss}I_s^2 \qquad U_s = 0 \text{ pro } I_s = 0$$

3) **energie více smyček**

$$U = \frac{1}{2}\sum_{kl} L_{kl} I_k I_l = \frac{1}{2}\sum_k I_k \Psi_k$$

• ukážeme ordenkciov:

I) pre 1 smyčku

II) $\Psi_s = L_{ss} I_s + \sum_\ell L_{s\ell} I_\ell$

$\Psi_k = L_{ks} I_s + \sum_\ell' L_{k\ell} I_s$

$\frac{dI_k}{dt} = 0 \qquad \frac{dI_s}{dt} \neq 0$

$$\frac{dA}{dt} = I_s \frac{d\Psi_s}{dt} + \sum_k' I_k \frac{d\Psi_k}{dt} = L_{ss} I_s \frac{dI_s}{dt} + \sum' I_k L_{ss} \frac{dI_s}{dt}$$

$\Big\}$ integrace $\int_0^{I_s} dI$

$$\Delta A = \frac{1}{2} L_{ss} I_s^2 + \sum_k' L_{ks} I_k I_s$$

$$U_{sys.smyček+s} = U_{sys.smyček} + \Delta A$$

IP

$$= \frac{1}{2} \sum_{k\ell}' L_{k\ell} I_k I_\ell + \frac{1}{2} \sum_k L_{ks} I_k I_s$$
$$+ \frac{1}{2} \sum_k' L_{sk} I_s I_k + \frac{1}{2} L_{ss} I_s^2$$

Magnetostatická energie

Elektrická energie – súhrn:

$U_E = \frac{1}{2} Q_k \Phi_k = \frac{1}{2} C_{k\ell} \phi_k \phi_\ell$ → systém vodičov

$U_E = \frac{\varepsilon_0}{2} \int E^2 dV = \frac{1}{2} \int \varrho \phi \, dV$ → spojité rozloženie

$u_E = \frac{\varepsilon_0}{2} E^2$

Magnetická energie

$U_M = \frac{1}{2} I_k \Psi_k = \frac{1}{2} L_{k\ell} I_k I_\ell$ → systém smyček

$U_M = \frac{\varepsilon_0 c^2}{2} \int B^2 dV = \frac{1}{2} \int \vec{j} \cdot \vec{A} \, dV$ → spojité prúdy

$u_M = \frac{1}{2\mu_0} B^2$

→ Dk: $U_M = \frac{\varepsilon_0 c^2}{2} \int B^2 dV = \frac{\varepsilon_0 c^2}{2} \int \vec{B} \cdot (\vec{\nabla} \times \vec{A}) \, dV = \frac{\varepsilon_0 c^2}{2} \int \vec{A} \cdot (\nabla \times \vec{B}) + \nabla \cdot (\vec{A} \times \vec{B}) dV$

$$= \frac{\varepsilon_0 c^2}{2} \int \vec{A} \cdot \vec{j}\, dV + [\cancel{\text{okrajový člen v} \infty}]^{\to 0}$$

systém smyček:

$$U_M = \frac{1}{2} \sum_k \int_{\gamma_k} I_k \vec{A} \cdot d\vec{s}_k = \frac{1}{2} \sum_k I_k \int_{\gamma_k} \vec{A} \cdot d\vec{s} = \frac{1}{2} \sum_k I_k \int \nabla \times \vec{A} \cdot d\vec{S} = \frac{1}{2} \sum_k I_k \Psi_k$$

Lokální zákony zachování

veličina na konci − veličina na začátku

+ množství vel. která opustí oblast = množství veličiny co vznikla

$$\int_V w|_{t_1} dV - \int_V w|_{t_2} dV + \int_{t_1}^{t_2} \int_{\partial V} \vec{w} \cdot d\vec{S}\, dt = \int_{t_1}^{t_2} \int s\, dV dt$$

w ... hust. veličiny

$\vec{w}$ ... hust. toku $(\vec{w} = \vec{v} w)$

s ... hust. tvorby veličiny

diff. tvar:

$$\frac{\partial w}{\partial t} + \nabla \cdot \vec{w} = s \qquad w^\mu = \begin{bmatrix} cw \\ \vec{w} \end{bmatrix} \qquad \nabla_\mu w^\mu = s$$

Maxwellovy rovnice

(i) $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \varrho$ (iii) $\nabla \cdot \vec{B} = 0$

(ii) $\nabla \times \vec{E} = -\partial_t B$ (iv) $\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2} \partial_t \vec{E}$

Zachování náboje

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot \vec{j} = 0 \qquad \nabla \cdot (iv) = \nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{j} + \frac{1}{c^2} \partial_t \underbrace{\nabla \cdot \vec{E}}_{\varepsilon_0 \varrho}$$

$$\nabla_\mu j^\mu = 0 \qquad \Rightarrow 0 = \nabla \cdot \vec{j} + \partial_t \varrho$$

⇒ rovnice kontinuity vbudovaná do Maxwellích

Bilance energie

hust. výkonu

$$-w = \vec{j}\cdot\vec{E} = \varepsilon_0 c^2\left(-\nabla\times\vec{B} + \frac{1}{c^2}\partial_t\vec{E}\right)\cdot\vec{E} =$$

$$= \varepsilon_0 c^2\left(\nabla\cdot(\vec{E}\times\vec{B}) - \vec{B}\cdot\nabla\times\vec{E} + \frac{1}{2}\frac{1}{c^2}\partial_t E\right) =$$

$$= \frac{\partial}{\partial t}\left(\frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) + \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B})$$

$$\Rightarrow \quad -\vec{j}\cdot\vec{E} = \frac{\partial u}{\partial t} + \nabla\cdot\vec{S}$$

$$u = \frac{\varepsilon_0}{2}E^2 + \frac{\varepsilon_0 c^2}{2}B^2 = \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{H}\cdot\vec{B}$$ → hustota elmag energie

$$\vec{S} = \varepsilon_0 c^2\,\vec{E}\times\vec{B}$$ → Poyntingov vektor

Bilance hybnosti

hust. síly

$$-\vec{f} = -(\varrho\vec{E} + \vec{j}\times\vec{B}) = -\varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} + \varepsilon_0\,\partial_t\vec{E}\times\vec{B} =$$

$$= -\varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \varepsilon_0 c^2(\nabla\cdot\vec{B})\vec{B} + \varepsilon_0\,\partial_t(\vec{E}\times\vec{B}) + \varepsilon_0\vec{E}\times(\nabla\times\vec{E}) + \varepsilon_0 c^2\vec{B}\times(\nabla\times\vec{B}) =$$

$$= \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \varepsilon_0(\vec{E}\cdot\nabla)\vec{E} + \varepsilon_0(\nabla\vec{E})\cdot\vec{E} - \varepsilon_0 c^2(\nabla\cdot\vec{B})\vec{B} - \varepsilon_0 c^2(\vec{B}\cdot\nabla)\vec{B} + \varepsilon_0 c^2(\nabla\vec{B})\cdot\vec{B} =$$

$$= \frac{\partial}{\partial t}(\varepsilon_0\vec{E}\times\vec{B}) - \varepsilon_0\nabla\cdot\left[\vec{E}\otimes\vec{E} + c^2\vec{B}\otimes\vec{B} - \frac{1}{2}(E^2 + c^2B^2)\overleftrightarrow{\delta}\right]$$

$$\boxed{-\vec{f} = \frac{\partial}{\partial t}\vec{g} + \vec{\nabla}\cdot\overleftrightarrow{T}}$$

$$\vec{g} = \varepsilon_0\vec{E}\times\vec{B} = \vec{D}\times\vec{B} = \frac{1}{c^2}\vec{S}$$ → hustota hybnosti

$$\overleftrightarrow{T} = -\varepsilon_0\left[\vec{E}\otimes\vec{E} + c^2\vec{B}\otimes\vec{B} - \frac{1}{2}(E^2 + c^2B^2)\overleftrightarrow{\delta}\right]$$ → tenzor toku hybnosti

$$= -\overleftrightarrow{\varepsilon}$$ → Maxwellov tenzor napätí

$$T^{\alpha\beta}_{EM} = \begin{bmatrix} u & c\vec{g} \\ \frac{1}{c}\vec{S} & \overleftrightarrow{T} \end{bmatrix} \qquad \phi^{\alpha} = \begin{bmatrix} w/c \\ \vec{f} \end{bmatrix} = F^{\alpha}{}_{\beta} j^{\beta}$$

$$\nabla_{\mu} T^{\mu\nu}_{EM} = \phi^{\nu}$$

$$T^{\alpha\beta}_{EM} = \varepsilon_0 c^2 \left[F^{\alpha\mu} F^{\beta\nu} \eta_{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \eta^{\alpha\beta} \right]$$

Úplné řešení Maxwella

Maxwell ky

(1) $\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho$ (3) $\nabla \cdot \vec{B} = 0$

(2) $\nabla \times \vec{B} - \frac{1}{c^2} \partial_t \vec{E} = \frac{1}{\varepsilon_0 c^2} \vec{j}$ (4) $\nabla \times \vec{E} + \partial_t \vec{B} = 0$

Potenciály

$$\left.\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \end{aligned}\right\} \text{splňuje (3),(4)}$$

$$\frac{\rho}{\varepsilon_0} = \nabla \cdot \vec{E} = -\nabla^2\phi - \nabla \cdot \frac{\partial \vec{A}}{\partial t} = -\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right]\phi - \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}\right)$$

$$\frac{1}{\varepsilon_0 c^2} \vec{j} = \nabla \times (\nabla \times \vec{A}) + \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi = -\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right]\vec{A} + \vec{\nabla}\left(\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla \cdot \vec{A}\right)$$

$$\Box \phi + \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla \cdot \vec{A}\right) = -\frac{1}{\varepsilon_0}\rho \qquad \Box = \nabla_\mu \nabla_\nu \eta^{\mu\nu}$$

$$\Box \vec{A} - \nabla\left(\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla \cdot \vec{A}\right) = -\frac{1}{\varepsilon_0 \mu}\vec{j}$$

$$\Box A^{\mu} - \nabla^{\mu}\nabla_{\nu}A^{\nu} = -\frac{1}{\varepsilon_0 c^2} j^{\mu}$$

Kalibrační transformace

$\vec{A} \longrightarrow \vec{A}' = \vec{A} + \nabla\psi \qquad \vec{B}' = \vec{B}$

$\phi \longrightarrow \phi = \phi - \frac{\partial\psi}{\partial t} \qquad \vec{E}' = \vec{E}$

Lorentz

$\frac{1}{c^2}\frac{\partial\phi}{\partial t} + \nabla\cdot\vec{A} = 0 \qquad \nabla_\mu A^\mu = 0 \qquad \rightsquigarrow$ relat. inv.

Coulomb

$\nabla\cdot\vec{A} = 0 \leftsquigarrow$ závisí na IS

$\Rightarrow \nabla^2\phi = -\frac{1}{\varepsilon_0}\rho \qquad \rightsquigarrow$ akauzální řešení pro ϕ

$\Box\vec{A} = -\frac{1}{\varepsilon_0 c}\vec{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi$

2 stupně volnosti pro EM pole

Weyl

$\phi = \phi_{cokoliv} \Rightarrow \vec{A}$ je pain

Řešení nehomog. vlnové rovnice

$\Box A^\mu = -\frac{1}{\varepsilon_0 c^2} j^\mu \qquad \nabla_\mu A^\mu = 0$

↳ **Greenova funkce:**

$-\Box G(x|x') = \delta(x|x') \rightarrow$ potom $A^\mu(x) = \frac{1}{\varepsilon_0 c^2}\int_M G(x|x') j^\mu(x')\, d\Omega'$

(homog. řešení

$\Box A^\mu = 0$

Požadavky na Greenovou fci:

(i) $G(x|x') = G(\Delta \mathbf{x})$ → translační invariance

(ii) $G(x|x') = G(\Delta \mathbf{x}^2)$ → rotačně & boost. symetrie

(iii) EM se šíří světelně

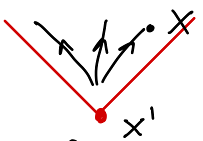
(iv) tenzorový charakter Greena: $G^{\mu}{}_{\nu}(x|x') = G(x|x')\,\delta^{\mu}_{\nu}$

} plochý p.č. Minkowsky

pole pouze "po" zdroji → retardované pole (Greenova fce)

pole pouze "před" zdrojem → advancované pole (Greenova fce)

Retardované



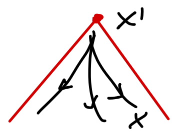
$G_{ret}(x|x') =$

$= 2\Theta(\Delta t)\, G_{sym}(x|x')$

$= \frac{1}{2\pi}\Theta(\Delta t)\,\delta(\Delta x^2)$

↳ nulové mimo budoucí kužel

Advancované



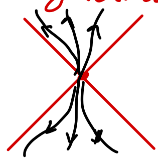
$G_{adv}(x|x') =$

$= 2\Theta(-\Delta t)\, G_{sym}(x|x')$

$= \frac{1}{2\pi}\Theta(-\Delta t)\,\delta(\Delta x^2)$

↳ nulové mimo minulý kužel

Symetrické



$G_{sym}(x|x') =$

$= \frac{1}{2}\left(G_{adv}(x|x') + G_{ret}(x|x')\right)$

$G_{sym}(x|x') = \frac{1}{4\pi}\delta(\Delta x^2)$

(iii) + (ii)

$$\delta(\Delta x^2) = \delta(c^2\Delta t^2 - r^2) = \left| \begin{array}{l} f(c\Delta t) = 0 \\ c\Delta t = \pm r \\ |f'(\pm r)| = 2r \end{array} \right| = \frac{1}{2r}\delta(c\Delta t - r) + \frac{1}{2r}\delta(c\Delta t + r)$$

$$\delta(f(x)) = \sum_{\substack{x_0 \in \mathbb{R} \\ f(x_0)=0}} \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

$$\Rightarrow G_{sym}(x|x') = \frac{1}{8\pi r}\delta(c\Delta t - r) + \frac{1}{8\pi r}\delta(c\Delta t + r)$$

$$G_{ret}(x|x') = \frac{1}{2\pi r}\delta(c\Delta t - r)$$

$$G_{adv}(x|x') = \frac{1}{2\pi r}\delta(c\Delta t + r)$$

Odvodenie Greenovej fce

$$-\Box G = \delta \qquad \Big/ \frac{1}{(2\pi)^2}\int \dots \exp(-i k_\mu \Delta x^\mu)\, d^4\Delta x$$ ← FT

$$k^\mu = \begin{bmatrix} \omega/c \\ \vec{k} \end{bmatrix} \qquad \Delta x^\mu = \begin{bmatrix} c\Delta t \\ \vec{r} \end{bmatrix}$$

LS:

$\eta^{\mu\nu}\nabla_\mu\nabla_\nu$

$$-\frac{1}{(2\pi)^2}\int \Box G \exp(-i k_\mu \Delta x^\mu)\, d^4\Delta x = -\frac{1}{(2\pi)^2}\int G \underbrace{\Box \exp(-i k_\mu \Delta x^\mu)}_{k_\mu k^\mu \exp(\dots)} d^4\Delta x$$

$$= k^2 \tilde{G}(k_\mu)$$

PS:

$$\frac{1}{(2\pi)^2}\int \delta \exp(-i k_\mu \Delta x^\mu)\, d^4\Delta x = \frac{1}{(2\pi)^2}$$

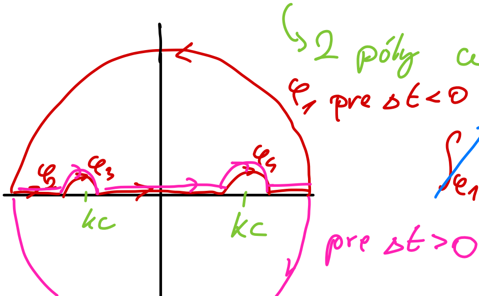
$$\Rightarrow \quad \tilde{G}(k_\mu) = \frac{1}{(2\pi)^2 k^2} = \frac{1}{(2\pi)^2\left(k^2 - \frac{\omega^2}{c^2}\right)}$$

$$\Rightarrow \quad G(k_\mu) = \frac{1}{(2\pi)^2}\int \tilde{G}(k_\mu)\exp(i k_\mu \Delta x^\mu)\, d^4\Delta x =$$

$$= \frac{1}{(2\pi)^4}\int \frac{\exp(i k_\mu \Delta x^\mu)}{k^2}\, d^4\Delta x =$$

$$= \frac{c}{(2\pi)^4}\iint \frac{1}{-\omega^2 + c^2k^2}\exp(-i\omega\Delta t)\, d\omega \, \exp(i\vec{k}\cdot\vec{r})\, d^3k$$

↳ 2 póly $\omega = \pm kc$



$$\underbrace{\int_{\varphi_1}}_{0} + \underbrace{\int_{\varphi_2}}_{I} + \underbrace{\int_{\varphi_3}}_{-\pi i \,\mathrm{Res}(f, -kc)} + \underbrace{\int_{\varphi_4}}_{-\pi i\, \mathrm{Res}(f, kc)} = 0$$

$$I = -\int \frac{\exp(-i\omega\Delta t)}{\omega^2 - c^2k^2} = \pi i \left(\mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, -kc\right) + \mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, kc\right)\right)$$

$$= \pi i \left(\frac{e^{ikc\Delta t}}{2kc} - \frac{e^{-ikc\Delta t}}{2kc} \right) = \frac{-\pi}{kc} \sin(kc\Delta t) \qquad \Delta t < 0$$

$$= \pi i \left(\mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, -kc\right) + \mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, kc\right)\right)$$

$$-2\pi i \left(\mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, -kc\right) + \mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, kc\right)\right) =$$

$$= -\pi i \left(\mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, -kc\right) + \mathrm{Res}\left(-\frac{e^{-i\omega\Delta t}}{\omega^2 - c^2k^2}, kc\right)\right) = \frac{\pi}{kc} \sin(kc\Delta t)$$

$$\Rightarrow \quad \boxed{I = \frac{\pi}{kc} |\sin(kc\Delta t)|}$$

$$\Rightarrow G(x_\mu) = \frac{c}{(2\pi)^4} \int \frac{\pi}{kc} \sin(kc|\Delta t|) \exp(i\vec{k}\cdot\vec{r})\, d^3k =$$

$$= \frac{1}{2} \frac{1}{(2\pi)^3} \int \frac{\sin(kc|\Delta t|)}{k} \exp(i\vec{k}\cdot\vec{r}) \underbrace{d^3k}_{k^2 \sin\vartheta\, dk\, d\vartheta\, d\varphi} =$$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty}$$

$$= \frac{\pi}{(2\pi)^4} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} k \sin(kc|\Delta t|)\, e^{ikr\overbrace{\cos\vartheta}^{-\xi}}\, dk\, \overbrace{\sin\vartheta\, d\vartheta}^{d\xi}\, d\varphi =$$

$$= \frac{\pi}{(2\pi)^4} \int_0^{2\pi} \int_{-1}^{1} \int_0^{\infty} k \sin(kc|\Delta t|)\, e^{-ikr\xi}\, dk\, d\xi\, d\varphi =$$

$$= \frac{1}{2} \frac{1}{(2\pi)^2} \int_0^{\infty} k \sin(kc|\Delta t|) \underbrace{\int_{-1}^{1} e^{-ikr\xi}\, d\xi}_{\frac{1}{ikr}\left(e^{ikr\xi} - e^{-ikr\xi}\right)}\, dk =$$

$$= \frac{1}{2}\frac{1}{(2\pi)^2}\int_0^\infty \frac{\cancel{k}\sin(kc|\Delta t|)}{} \frac{\left(e^{ikr}-e^{-ikr}\right)}{i\cancel{k}r}\,dk =$$

$$= \frac{1}{r}\frac{1}{(2\pi)^2}\int_0^\infty \sin(kc|\Delta t|)\sin(kr)\;dk =$$

$$= \frac{1}{8\pi r}\delta(c|\Delta t|-r) \;=\; \begin{cases} \frac{1}{8\pi r}\delta(c\Delta t-r) & \Delta t>0 \\ \frac{1}{8\pi r}\delta(c\Delta t+r) & \Delta t<0 \end{cases}$$

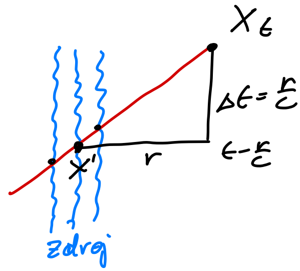
$$\Rightarrow \quad G_{sym} = \frac{1}{8\pi r}\left(\underbrace{\delta(c\Delta t-r)}_{\text{ret}} + \underbrace{\delta(c\Delta t+r)}_{\text{adv.}}\right)$$

Retardovaný potenciál

$$A^\mu(x) = \int G_{ret}(x|x')\,\frac{1}{\varepsilon_0 c^2}\, j^\mu(x')\, d^4\Omega =$$

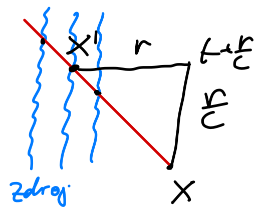
$$= \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{1}{r}\,\delta(c\underbrace{\Delta t}_{t-t'} - \underbrace{r}_{|\vec{x}-\vec{x}'|})\, j^\mu(t',\vec{x}')\, c\,dt'\, d^3\vec{x}' =$$

$$\boxed{\begin{aligned}
A^\mu &= \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{j^\mu(t-\frac{r}{c},\vec{x}')}{r}\, d^3\vec{x}' \\
\phi_{ret}(t,\vec{x}) &= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(t-\frac{r}{c},\vec{x}')}{r}\, d^3\vec{x}' \\
\vec{A}_{ret}(t,\vec{x}) &= \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{\vec{j}(t-\frac{r}{c},\vec{x}')}{r}\, d^3\vec{x}'
\end{aligned}}$$



$$\phi_{adv}(t,\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(t+\frac{r}{c},\vec{x}')}{r}\, d^3\vec{x}'$$

$$A_{adv}(t,\vec{x}) = \frac{1}{4\pi\varepsilon_0 c^2}\int \frac{\vec{j}(t+\frac{r}{c},\vec{x}')}{r}\, d^3\vec{x}'$$



Jefimenkovy vztahy pro E a B

$$\vec{E}(t,\vec{x}) = -\nabla\phi - \frac{\partial}{\partial t}\vec{A} =$$

$$\nabla\left(t-\frac{r}{c}\right) = -\frac{1}{c}\vec{e}_r$$

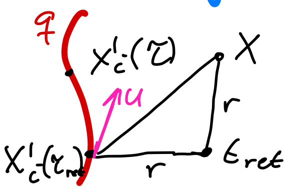
$$= -\frac{1}{4\pi\varepsilon_0}\int d^3\vec{x}'\left(\nabla\frac{\rho(t-\frac{r}{c},\vec{x}')}{r} + \frac{1}{c^2}\frac{\partial}{\partial t}\frac{\vec{j}(t+\frac{r}{c},\vec{x}')}{r}\right) =$$

$$= -\frac{1}{4\pi\varepsilon_0}\int d^3\vec{x}'\left(-\frac{1}{c}\frac{\partial\rho}{\partial t}(t-\frac{r}{c},\vec{x}')\frac{\vec{e}}{r} - \rho(t-\frac{r}{c},\vec{x}')\frac{\vec{e}}{r^2} - \frac{1}{c^2}\frac{\partial\vec{j}}{\partial t}(t-\frac{r}{c},\vec{x}')\frac{1}{r}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\int d^3\vec{x}'\Big(\underbrace{\frac{\rho(t-\frac{r}{c},\vec{x}')}{r^2}}_{\text{Coulombický člen}}\vec{e} + \underbrace{\frac{\partial\rho}{\partial t}(t-\frac{r}{c},\vec{x}')\frac{\vec{e}}{cr} - \frac{\partial\vec{j}}{\partial t}(t-\frac{r}{c},\vec{x}')\frac{1}{c^2 r}}_{\text{zářivé členy}}\Big)$$

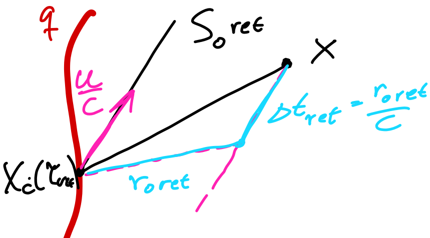
$$\vec{B}(t,\vec{x}) = \nabla\times\vec{A} =$$
$$= \frac{1}{4\pi\varepsilon_0 c^2}\int d^3\vec{x}\Big[\underbrace{\vec{j}\big(t-\tfrac{r}{c},\vec{x}'\big)\times\frac{\vec{e}}{r^2}}_{\text{Biot-Savart}} + \underbrace{\frac{\partial\vec{j}}{\partial t}\big(t-\tfrac{r}{c},\vec{x}'\big)\times\frac{\vec{e}}{cr}}_{\text{zářivé členy}}\Big]$$

Potenciály bodového zdroje



$$j^\mu = \int q\, u^\mu(\tau)\,\delta(X|X_c(\tau))\, c\, d\tau$$
$$j^\mu = q\, u^\mu(\tau_{ret})$$

$$A^\mu(X) = \int G_{ret}\frac{1}{\varepsilon_0 c^2} j^\mu(X')\, d^4\Omega = \frac{1}{2\pi\varepsilon_0}\iint \Theta(\Delta t)\,\delta((X-X')^2)\, c\, q\, u^\mu(\tau)\,\delta(X'|X_c(\tau))\, d\tau\, d^4\Omega'$$



$$= \frac{qc}{4\pi\varepsilon_0 c^2}\int \Theta(\Delta t)\,\delta((X-X_c(\tau))^2)\, u^\mu(\tau)\, d\tau =$$
$$= \frac{qc}{4\pi\varepsilon_0 c^2}\int \Theta(\Delta t)\frac{1}{\left|\frac{df}{d\tau}(\tau_{ret})\right|}\,\delta(\tau-\tau_{ret})\, u^\mu(\tau)\, d\tau$$

$$= \left| \begin{array}{l} f'(\tau) = -2(x^\alpha - x^\alpha_c(\tau))\,\eta_{\alpha\beta}\, u^\beta(\tau) = -2c^2\Delta t_{0ret} = -2c\, r_{0ret} \\ f(\tau) = (x^\alpha - x^\alpha_c(\tau))\,\eta_{\alpha\beta}\,(x^\beta - x^\beta_c(\tau)) \end{array} \right| = \frac{q}{4\pi\varepsilon_0 c^2}\frac{u^\mu}{r_0}\Big|_{\tau=\tau_{ret}} =$$

$$r_0 = -\frac{1}{c}u_\alpha(x^\alpha - x^\alpha_c(\tau)) =$$
$$= -\Big[-\gamma, \gamma\frac{\vec{V}}{c}\Big]\cdot\begin{bmatrix} c\Delta t \\ \vec{r}\end{bmatrix}_{\tau=\tau_{ret}}$$
$c\Delta t = r|_{\tau=\tau_{ret}}$
$$= \gamma r\Big(1-\frac{\vec{e}\cdot\vec{V}}{c}\Big)$$

$$\Rightarrow \phi(t,\vec{x}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r\big(1-\frac{\vec{e}\cdot\vec{V}}{c}\big)}\Bigg|_{\tau=\tau_{ret}}$$
$$\vec{A}(t,\vec{x}) = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\vec{V}}{r\big(1-\frac{\vec{e}\cdot\vec{V}}{c}\big)}\Bigg|_{\tau=\tau_{ret}}$$

Liénard-Wiechertovy potenciály

Eldyn bez zdrojů

$$\nabla\cdot\vec{E} = 0 \qquad \nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{E} + \partial_t\vec{B} = 0 \qquad \nabla\times\vec{B} - \frac{1}{c^2}\partial_t^2\vec{E} = 0$$

$$\vec{E} \to c\vec{B} \qquad c\vec{B} \to -\vec{E}$$

$$0 = -\frac{1}{c^2}\partial_t^2\vec{E} + \nabla\times\partial_t\vec{B} = -\frac{1}{c^2}\partial_t^2\vec{E} + \Delta\vec{E} - \nabla\overbrace{\nabla\cdot E}^{0}$$

$$\Rightarrow \quad \Box\vec{E} = 0 \qquad \Box\vec{B} = 0$$

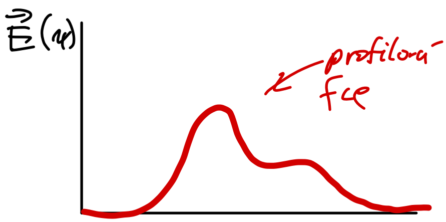
Rovinná vlna

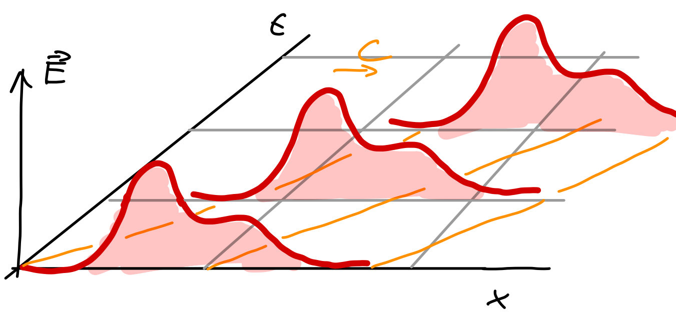
→ šíření jedným směrem $\vec{e}_\parallel$ $\quad r_\parallel = \vec{r}\cdot\vec{e}_\parallel$

$$\vec{E}(t, r_\parallel) = \vec{E}(k r_\parallel - \omega t)$$

$$\nabla\vec{E} = \underbrace{k\vec{e}_\parallel}_{\vec{k}}\vec{E}' \qquad \frac{\partial}{\partial t}\vec{E} = -\omega\vec{E}'$$

$$\Box\vec{E} = -\frac{\omega^2}{c^2}\vec{E}'' + k^2\vec{E}'' = 0 \quad\Rightarrow\quad \frac{\omega^2}{c^2} = k^2 \qquad \omega = ck$$

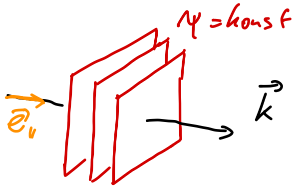




$$\psi = \text{konst} \Rightarrow r_\parallel = \text{konst} + \underbrace{\frac{\omega}{k}}_{c} t$$

$$\psi = k r_\parallel - \omega t = k_\mu x^\mu$$

$$x^\mu = \begin{bmatrix} ct \\ \vec r \end{bmatrix} \qquad k^\mu = \begin{bmatrix} \frac{\omega}{c} \\ \vec k \end{bmatrix} \qquad \psi = k_\mu x^\mu = \vec k \cdot \vec r - \omega t$$



$$\nabla\cdot\vec E = 0 \Rightarrow \vec k\cdot\vec E' = 0 \Rightarrow \vec k\cdot\vec E = 0 \Rightarrow \vec E \perp \vec k$$

$$\nabla\cdot\vec B = 0 \rightarrow \vec k\cdot\vec B' = 0 \Rightarrow \vec k\cdot\vec B = 0 \Rightarrow \vec B \perp \vec k$$

$$\nabla\times\vec E + \partial_t \vec B = 0 \Rightarrow c\vec k\times\vec E' - \omega c\vec B = 0 \Rightarrow c\vec B = \vec e_\parallel\times\vec E$$

$$\vec B\perp\vec E,\ E = cB$$

$$\nabla\times\vec B - \frac{1}{c^2}\partial_t\vec E = 0 \Rightarrow c\vec k\times c\vec B' + \omega\vec E' = 0 \Rightarrow \vec E = -\vec e_\parallel\times c\vec B$$

$$\tilde{\mathcal L} \propto \vec E\cdot c\vec B = 0$$

$$\mathcal L \propto E^2 - c^2B^2 = 0$$

$$u = \frac{\varepsilon_0}{2}E^2 + \frac{\varepsilon_0 c^2}{2}B^2 = \varepsilon_0 E^2 = \varepsilon_0 c^2 B^2$$

$$\vec P = \varepsilon_0 c^2\vec E\times\vec B = \varepsilon_0 E^2 c\vec e_\parallel = uc\vec e_\parallel$$

Monochromatické vlny

$\vec E(\psi)$ $\quad\vec E_0\cos\psi$ $\quad\vec E_0\sin\psi$

$\vec E = \mathrm{Re}\,\vec{\mathcal E}$ $\quad\vec B = \mathrm{Re}\,\vec{\mathcal B}$

$\vec{\mathcal E} = \vec{\mathcal E}_0\exp(i\psi) = \vec{\mathcal E}_0\exp(i\vec k\cdot\vec r - i\omega t)$

$\vec{\mathcal B} = \vec{\mathcal B}_0\exp(i\psi) = \vec{\mathcal B}_0\exp(i\vec k\cdot\vec r - i\omega t)$

$\vec k\cdot\vec{\mathcal E}_0 = \vec k\cdot\vec{\mathcal B}_0 = 0$

$c\vec{\mathcal B}_0 = \vec e_\parallel\times\vec{\mathcal E}_0$ $\quad\vec{\mathcal E}_0 = -\vec e_\parallel\times\vec{\mathcal B}_0$

$\vec{\mathcal E}_0 = E_1\exp(i\delta_1)\vec e_1 + E_2\exp(i\delta_2)\vec e_2$

Sférické vlny

Coulombická kalibrace $\vec{\nabla}\cdot\vec{A}=0$

$$\rho=0 \quad \Delta\phi=\frac{1}{\varepsilon_0}\rho=0 \quad \phi=0$$

$$\Rightarrow \quad \Box\vec{A}=0 \qquad \vec{E}=-\partial_t\vec{A} \qquad \vec{B}=\nabla\times\vec{A}$$

Debyeův potenciál: $\vec{A}=\vec{\mathbb{L}}\,\psi \quad \rightarrow \quad \vec{\mathbb{L}}=-i\,\vec{r}\times\vec{\nabla}$

$$\vec{r}\cdot\vec{\mathbb{L}}=0 \qquad \vec{\mathbb{L}}\,r=0 \qquad \vec{\mathbb{L}}\cdot\vec{\mathbb{L}}=\mathbb{L}^2=\Delta_{S_2}=\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\mathbb{L}^2$$

$$\Delta\mathbb{L}=\mathbb{L}\Delta \quad \Rightarrow \quad \Box\mathbb{L}=\mathbb{L}\Box$$

$$\vec{e}_z\cdot\mathbb{L}=-i\frac{\partial}{\partial\varphi} \qquad \vec{\mathbb{L}}\times\vec{\mathbb{L}}=i\vec{\mathbb{L}} \qquad [\vec{a}\cdot\vec{\mathbb{L}},\vec{b}\cdot\vec{\mathbb{L}}]=i(\vec{a}\times\vec{b})\cdot\vec{\mathbb{L}}$$

$\Rightarrow$ Vlnová rovnice:

$$\Box\vec{A}=\Box\vec{\mathbb{L}}\psi=\vec{\mathbb{L}}\,\Box\psi=0$$

$$\Box\psi=0 \quad \Rightarrow \quad \Box\vec{A}=0$$

TE - pole

$$\Box\psi^{TE}=0 \quad \rightarrow \quad \vec{E}=-\frac{\partial\vec{A}}{\partial t}=-\frac{\partial}{\partial t}\vec{\mathbb{L}}\psi^{TE} \quad \rightarrow \quad \vec{r}\cdot\vec{E}=0$$

$$\vec{B}=\nabla\times\vec{A}=\nabla\times\vec{\mathbb{L}}\psi^{TE} \quad \rightarrow \quad \vec{r}\cdot\vec{B}=\vec{r}\cdot\nabla\times\vec{\mathbb{L}}\psi^{TE}=$$

$$=\vec{r}\times\vec{\nabla}\cdot\vec{\mathbb{L}}\psi^{TE}=i\,\mathbb{L}^2\psi^{TE}$$

TM - pole

$$\vec{E}\rightarrow c\vec{B} \qquad c\vec{B}\rightarrow-\vec{E}$$

$$\Rightarrow \quad \vec{E}=\nabla\times\vec{A}_E \qquad \vec{A}_E=\vec{\mathbb{L}}\psi^{TM} \quad \Rightarrow \quad \vec{r}\cdot\vec{B}=0$$

$$\vec{r}\cdot\vec{E}=ic\,\mathbb{L}^2\psi^{TE}$$

Vlnová rovnice

$$\Box\psi = 0 \quad \to \quad \psi = R(r)\,Y(\vartheta,\varphi)\,\mathcal{E}(t)$$

$$\frac{1}{\psi}\Box\psi = 0 = \underbrace{-\frac{1}{c^2}\frac{1}{\mathcal{E}}\frac{d^2\mathcal{E}}{dt^2}}_{-\omega^2} + \underbrace{\frac{1}{r^2}\frac{1}{R}\frac{d}{dr}r^2\frac{dR}{dr}} + \underbrace{\frac{1}{r^2}\frac{1}{Y}\mathbb{L}^2 Y}_{-\ell(\ell+1)} = 0$$

$$\Rightarrow \frac{d^2\mathcal{E}}{dt^2} + \omega^2\mathcal{E} = 0 \quad \to \quad \mathcal{E}(t) = \exp(i\omega t)$$

$$-\mathbb{L}^2 Y = \ell(\ell+1)Y \quad \Rightarrow \quad Y(\vartheta,\varphi) = Y_\ell^m(\vartheta,\varphi)$$

$$\left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + k^2 - \frac{\ell(\ell+1)}{r^2}\right]R = 0 \quad \to \text{ sferické Besselovy fce}$$

$$R_{k\ell}(r) = \begin{cases} j_\ell(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{\ell+\frac{1}{2}}(kr) \\ n_\ell(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} N_{\ell+\frac{1}{2}}(kr) \end{cases}$$

$$j_\ell(\xi) = (-\xi)^\ell\left[\frac{1}{\xi}\frac{d}{d\xi}\right]^\ell\frac{\sin\xi}{\xi} \approx \frac{\xi^\ell}{(2\ell+1)!!} \qquad \xi \ll 1$$

$$n_\ell(\xi) = (-\xi)^\ell\left[\frac{1}{\xi}\frac{d}{d\xi}\right]^\ell\frac{\cos\xi}{\xi} \approx \frac{(2\ell-1)!!}{\xi^{\ell+1}} \qquad \xi \gg 1$$

$$\Rightarrow \psi_{k\ell m} = R_{k\ell}(r)\,Y_\ell^m(\vartheta,\varphi)\,e^{-i\omega t} \qquad k\in\mathbb{R}^+,\ \ell\in\mathbb{N}_0,\ m=-\ell,\dots,\ell$$

Plné řešení

$$\Box \vec{A} = 0 \iff \vec{A} = \vec{\mathbb{L}}\,\psi \qquad \Box \psi = 0$$

$$\vec{A}^{TE}_{klm} = \vec{\mathbb{L}}\,\psi^{TE}_{klm} \qquad \vec{A}^{TM}_{klm} = \vec{\mathbb{L}}\,\psi^{TM}_{klm}$$

→ monochromatické ř.

$$\frac{1}{c}\vec{E} = \sum_{lm} \Big(-a^{TE}_{klm} \overbrace{\frac{1}{c}\partial_t}^{-ik} \vec{\mathbb{L}}\,\psi^{TE}_{klm} + a^{TM}_{klm} \nabla\times\vec{\mathbb{L}}\,\psi^{TM}_{klm} \Big) =$$

$$\vec{B} = \sum_{lm}' \Big(a^{TE}_{klm} \nabla\times\vec{\mathbb{L}}\,\psi^{TE}_{klm} + a_{klm}^{TM} \overbrace{\frac{1}{c}\frac{\partial}{\partial t}}^{ik} \vec{\mathbb{L}}\,\psi^{TM}_{klm}$$

$$\frac{1}{c}\vec{r}\cdot\vec{E} = \sum' a^{TM}_{klm}\, i\,\mathbb{L}^2 R^{TM}_{kl} Y_l^m e^{-i\omega t} = \sum' -i\,l(l+1)\, a^{TM}_{klm} R^{TM}_{kl} e^{-i\omega t} Y_l^m$$

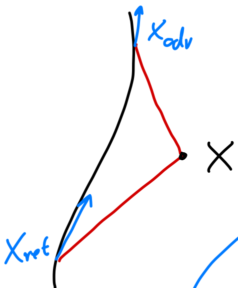
$$\vec{r}\cdot\vec{B} = \quad \dots \quad = \sum -i\,l(l+1)\, a^{TE}_{klm} R^{TE}_{kl} e^{-i\omega t} Y_l^m$$

$$\Rightarrow \int_{S_2} \frac{1}{c}\vec{r}\cdot\vec{E}\, d^2\Omega = -i\,l(l+1)\, a^{TM}_{klm} R^{TM}_{kl} e^{-i\omega t}$$

$$\int_{S_2} \vec{r}\cdot\vec{B}\, d\Omega = -i\,l(l+1)\, a^{TE}_{klm} R^{TE}_{kl} e^{-i\omega t}$$

Pole bodového zdroje

$$\boxed{\begin{aligned}\phi(t,\vec{x}) &= \frac{q}{4\pi\varepsilon_0}\,\frac{1}{r\left(1-\frac{\vec{e}\cdot\vec{v}}{c}\right)}\Bigg|_{t=t_{ret}}\\ \vec{A}(t,\vec{x}) &= \frac{q}{4\pi\varepsilon_0}\,\frac{\vec{v}}{r\left(1-\frac{\vec{e}\cdot\vec{v}}{c}\right)}\Bigg|_{t=t_{ret}}\end{aligned}}\quad\Bigg\}\ \text{Liénard-Wiechertovy potenciály}$$



$$(X - X_i(t_{ret}))^2 = 0 \qquad t_{ret} < t$$

$$c(t - t_{ret}) = |\vec{x} - \vec{x}_i(t_{ret})| = r_r$$

$$t_{ret}(t, x)$$

$\frac{\partial}{\partial t}$

$\vec{\nabla}$

$$-\vec{\nabla} c\, t_{ret} = \frac{1}{r_{ret}}\vec{\nabla}(x - \vec{x}_i(t_{ret}))\cdot \vec{r}_{ret}$$

$$= \vec{e}_{ret} - \vec{e}_{ret}\cdot\vec{v}\,\vec{\nabla} t_{ret}$$

$$1 - \frac{\partial t_{ret}}{\partial t} = -\frac{1}{c}\underbrace{\frac{\vec{r}_{ret}}{r_{ret}}}_{\vec{e}_{ret}}\,\overbrace{\frac{d\vec{x}_i}{dt}(t_{ret})}^{\vec{v}(t_{ret})}\,\frac{\partial t_{ret}}{\partial t}$$

$$\Rightarrow \frac{\partial t_{ret}}{\partial t} = \left(1 - \frac{\vec{e}_{ret}\cdot\vec{v}_{ret}}{c}\right)^{-1} \qquad \Rightarrow \vec{\nabla} c\, t_{ret} = -\frac{\vec{e}_{ret}}{1 - \frac{\vec{e}_{ret}\cdot\vec{v}}{c}}$$

$$\frac{\partial r_{ret}}{\partial t} = \frac{\partial}{\partial t}(ct - ct_{ret}) = \frac{-\vec{e}_{ret}\cdot\vec{v}_{ret}}{1 - \frac{\vec{e}_{ret}\cdot\vec{v}_{ret}}{c}}$$

$$\vec{\nabla} r_{ret} = \frac{\vec{e}_{ret}}{1 - \frac{\vec{e}_r\cdot\vec{v}_r}{c}}$$

$$\vec{\nabla}\vec{r}_{ret} = \vec{\nabla}(\vec{x} - \vec{x}_i(t_{ret})) = \overleftrightarrow{I} + \frac{\vec{e}_{ret}}{1 - \frac{\vec{e}_{ret}\cdot\vec{v}_{ret}}{c}}\,\frac{\vec{v}_{ret}}{c}$$

$$\vec{\nabla}\vec{v}_{ret} = (\vec{\nabla} t_{ret})\,\vec{a}_{ret} = -\frac{1}{c}\,\frac{\vec{e}_{ret}\,\vec{a}_{ret}}{1 - \frac{\vec{e}_{ret}\cdot\vec{v}_{ret}}{c}}$$

$$\frac{1}{c}\vec\nabla(\vec r_{ret}\cdot\vec v_{ret}) = \frac{\vec v_{ret}}{c} + \frac{v_{ret}^2}{c^2}\,\frac{\vec e_{ret}}{1-\frac{\vec e_{ret}\cdot\vec v}{c}} - \frac{r_{ret}}{c^2}\,\frac{\vec e_{ret}\,\vec a_{ret}\cdot\vec e_{ret}}{1-\frac{\vec e_{ret}\cdot\vec v_{ret}}{c}}$$

$$\vec\nabla\left(r_{ret} - \frac{\vec r_{ret}\cdot\vec v_{ret}}{c}\right) =$$

$$= \frac{1}{1-\frac{\vec e_{ret}\cdot\vec v_{ret}}{c}}\left[\vec e_{ret}\overbrace{\left(1-\frac{v_{ret}^2}{c^2}\right)}^{\gamma_{ret}^{-2}} - \left(1-\frac{\vec e_{ret}\cdot\vec v_{ret}}{c}\right)\frac{\vec v_{ret}}{c} + \frac{r_{ret}}{c^2}\,\vec a_{ret}\cdot\vec e_{ret}\,\vec e_{ret}\right]$$

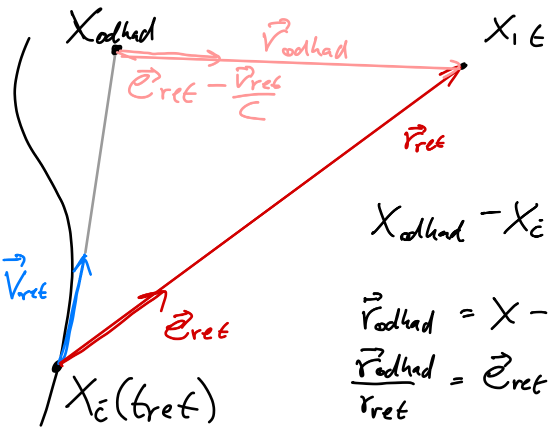
$$\frac{\partial}{\partial t}\left(r_{ret} - \frac{\vec r_{ret}\cdot\vec v_{ret}}{c}\right) = \frac{c}{1-\frac{\vec e_{ret}\cdot\vec v_{ret}}{c}}\left[-\frac{\vec e_{ret}\cdot\vec v_{ret}}{c} + \frac{v_{ret}^2}{c} - \frac{r_{ret}}{c^2}\,\vec a_{ret}\cdot\vec e_{ret}\right]$$

$$\vec E = -\nabla\phi - \frac{\partial\vec A}{\partial t} =$$

$$\boxed{\vec E = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{1-\frac{\vec e_{ret}\cdot\vec v_{ret}}{c}}\right)^3\left[\underbrace{\frac{1}{r_{ret}\,\gamma_{ret}^2}\left(\vec e_{ret}-\frac{\vec v_{ret}}{c}\right)}_{\text{rýchlostný/Coulombový člen}} + \underbrace{\frac{1}{c^2}\frac{1}{r_{ret}}\,\vec e_{ret}\times\left(\left(\vec e_{ret}-\frac{\vec v_{ret}}{c}\right)\times\vec a_{ret}\right)}_{\text{žiarivý člen}}\right]}$$

$$\boxed{\vec B = \frac{1}{c}\,\vec e_{ret}\times\vec E}$$

prostorový obr:



$$X_{odhad} = X_č(t_{ret}) + (t-t_{ret})\,\vec v_{ret}$$

$$\vec r_{odhad} = X - X_{odhad} = \vec r_{ret} - r_{ret}\frac{\vec v_{ret}}{c}$$

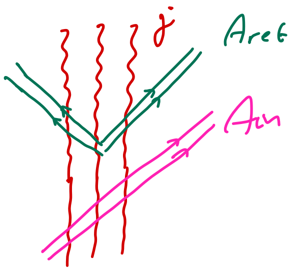
$$\frac{\vec r_{odhad}}{r_{ret}} = \vec e_{ret} - \frac{\vec v_{ret}}{c}$$

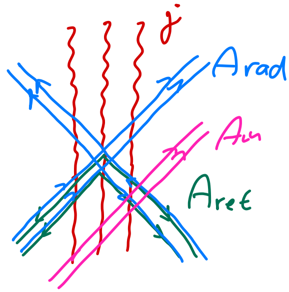
Feynmanov tvar

$$\vec E = \frac{q}{4\pi\varepsilon_0}\left[\frac{\vec e_{ret}}{r_{ret}^2} + \frac{r_{ret}}{c}\frac{d}{dt}\frac{\vec e_{ret}}{r_{ret}^2} + \frac{1}{c^2}\frac{d^2}{dt^2}\vec e_{ret}\right]$$

$$-\Box A^\mu = \frac{1}{\varepsilon_0 c^2} j^\mu$$

$$A^\mu = A^\mu_{ret} + A^\mu_{in} \qquad \Box A^\mu_{in} = 0$$
$$= A^\mu_{adv} + A^\mu_{out} \qquad \Box A^\mu_{out} = 0$$





$$A_{rad} = A_{ret} - A_{adv}$$
$$A_{out} = A_{in} + A_{rad}$$

$$\Box A_{rad} = \Box A_{ret} - \Box A_{adv} = \frac{1}{\varepsilon_0 c^2} j - \frac{1}{\varepsilon_0 c^2} j = 0$$

Brzdné žiarenie

$$M \ddot{z} = q\, F \cdot \dot{z}$$
$$\nabla \cdot F = \frac{1}{\varepsilon_0 c^2} J$$
$$\nabla_{[} F_{]} = 0$$

$$J(x) = q \int \dot{z}\, \delta(z|x)\, d\tau + J_{IN}$$

zdroj brzdnej sily

$$F = F_{SELF} + F_{IN}$$

$$q F_{SELF} \cdot \dot{z} = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{1}{s}\ddot{z} + \frac{2}{3}\left(\dddot{z} - \ddot{z}^2 \dot{z}\right) + O(s)\right]$$

$$\underbrace{\left(M + \frac{q^2}{4\pi\varepsilon_0}\frac{1}{2}\frac{1}{s}\right)}_{m}\ddot{z} = \frac{q^2}{4\pi\varepsilon_0}\underbrace{\frac{2}{3}\left(\dddot{z} - \ddot{z}^2 \dot{z}\right)}_{\text{brzdná sila}} + q F_{IN} \dot{z}$$

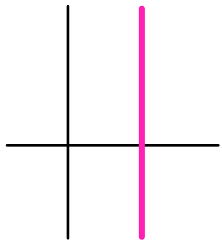
$$\left.\begin{aligned} a &= \tau_*(\dot a - a^2 u) \\ u &= \mathrm{ch}\beta\, e_\tau + \mathrm{sh}\beta\, e_z \end{aligned}\right\} \begin{aligned} a &= \dot\beta(\mathrm{sh}\beta\, e_t + \mathrm{ch}\beta\, e_z) \\ \dot a &= \ddot\beta(\mathrm{sh}\beta\, e_t + \mathrm{ch}\beta\, e_z) + \dot\beta^2(\mathrm{ch}\beta\, e_t + \mathrm{sh}\beta\, e_z) \end{aligned}$$

$$\Rightarrow \quad \dot a - a^2 u = \ddot\beta(\mathrm{sh}\beta\, e_t + \mathrm{ch}\beta\, e_z)$$

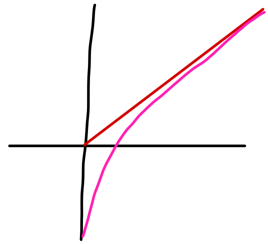
$$\dot\beta = \tau_* \ddot\beta$$

$$\hookrightarrow \dot\beta = a_0 \exp\frac{\tau}{\tau_*} \quad \rightarrow \quad \beta = a_0 \tau_* \exp\frac{\tau}{\tau_*} + \beta_\infty$$

$$t = \int \mathrm{ch}\beta\, d\tau$$
$$z = \int \mathrm{sh}\beta\, d\tau$$



$a_0 = 0$



$a_0 \neq 0$