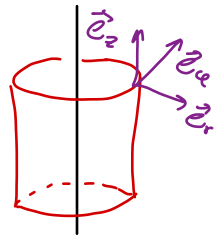
1

$$\vec{A} = \nabla(x^2+y^2-2z^2) = \nabla f = (2x, 2y, -4z) = 2x\vec{e}_x + 2y\vec{e}_y - 4z\vec{e}_z$$

$$\oint_{\partial V} \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot\vec{A}\, dV$$



$$\nabla\cdot\vec{A} = \nabla^2 f = 2+2-4 = 0$$

$$d\vec{S} = \pm \frac{\partial\vec{x}}{\partial u} \times \frac{\partial\vec{x}}{\partial v}\, du\, dv$$

$$\vec{x} : (u\ v) \to \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x = A\sin\varphi$$
$$y = A\cos\varphi$$
$$z = z$$

$$\frac{\partial\vec{x}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{\partial\vec{x}}{\partial\varphi} = \begin{pmatrix} A\cos\varphi \\ -A\sin\varphi \\ 0 \end{pmatrix}$$

$$-\frac{\partial\vec{x}}{\partial z} \times \frac{\partial\vec{x}}{\partial\varphi} = \begin{pmatrix} +A\sin\varphi \\ -A\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -A\sin\varphi \\ -A\cos\varphi \\ 0 \end{pmatrix} = \vec{n}$$

$$d\vec{S} = -A(\sin\varphi\,\vec{e}_x + \cos\varphi\,\vec{e}_y)\, d\varphi\, dz$$

$$\vec{A} = (2A\sin\varphi, 2A\cos\varphi, -4z)$$

$$\vec{A}\cdot d\vec{S} = -2A^2\sin^2\varphi - 2A^2\cos^2\varphi = -2A^2$$

$$\int_{plášť} \vec{A}\cdot d\vec{S} = \int_0^{2\pi}\int_{z_0}^{z_1} -2A^2 = -4\pi A^2(z_1-z_0) = 4\pi A^2(z_0-z_1)$$

$$\int_{podlop} \vec{A}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{A} 4z_1\ r\,dr\,d\varphi - \int_0^{2\pi}\int_0^{A} 4z_0\ r\,dr\,d\varphi =$$

$$= 4\pi A^2(z_1-z_0)$$

②

$$\vec{A} = \nabla(x^2+y^2-2z^2) = (2x, 2y, -4z) =$$

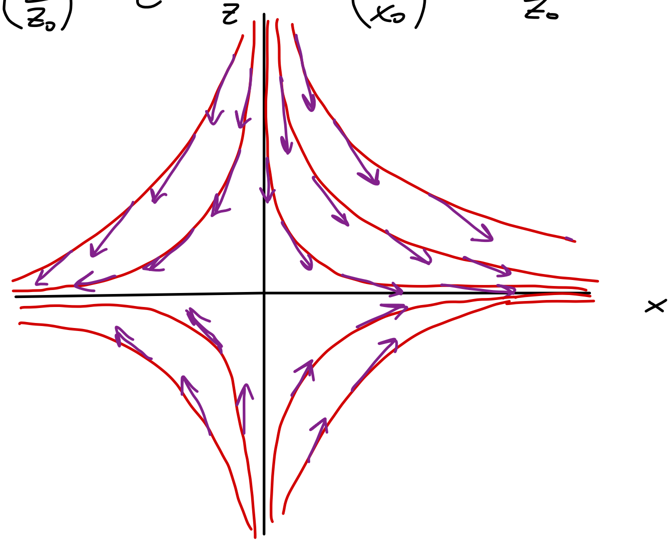
$$\frac{d}{ds}\begin{pmatrix}x(s)\\ z(s)\end{pmatrix} = \begin{pmatrix}2x(s)\\ -4z(s)\end{pmatrix} \qquad x' = 2x \qquad z' = -4z$$

$$x = x_0 e^{2s} \qquad z = z_0 e^{-4s}$$

$$\vec{x}(s) = x_0 e^{2s}\vec{e}_x + z_0 e^{-4s}\vec{e}_y \qquad \left(\frac{x}{x_0}\right)^{\frac{1}{2}} = \left(\frac{z}{z_0}\right)^{-\frac{1}{4}}$$

$$\Rightarrow \frac{z}{z_0} = \left(\frac{x}{x_0}\right)^{-2}$$

$$\sqrt{\frac{x}{x_0}} = e^s \qquad \left(\frac{z}{z_0}\right)^{-\frac{1}{4}} = e^s \qquad \left(\frac{x}{x_0}\right)^{-2} = \frac{z}{z_0}$$



③

$$\nabla f(|\vec{r}|) = \nabla f(r) = \frac{df}{dr}\nabla r$$

$$(\nabla r)_i = \partial_i (x_j x_j)^{\frac{1}{2}} = \frac{1}{2}(x_j x_j)^{-\frac{1}{2}}\, 2x_i = \frac{x_i}{r}$$

$$\nabla r = \frac{\vec{r}}{r} = \vec{e}_r$$

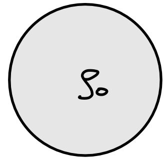
$$\Rightarrow \nabla f(|\vec{r}|) = f'\vec{e}_r$$

$$\nabla^2 f = \nabla\cdot\nabla f = \vec{e}_r\cdot\nabla f' + f'\nabla\cdot\vec{e}_r = f''\overbrace{\vec{e}_r\cdot\vec{e}_r}^{1} + f'\left(\frac{3}{r}+r\right) = f'' + \frac{2}{r}f'$$

$$\partial_i \frac{x_i}{r} = \frac{3}{r} + x_i\,\partial_i\frac{1}{r} = \frac{1}{r} - x_i\cdot\frac{x_i}{r^3} = \frac{3}{r} - \frac{1}{r}$$

$$\nabla^2 f = f'' + \frac{2}{r}f' = \frac{1}{r^2}\left(r^2 f'\right)' = \frac{1}{r}(rf)''$$

④



$$\nabla \cdot \vec{E} = \frac{\varrho_0}{\varepsilon_0} \quad \rightarrow \quad \oint \vec{E} \cdot d\vec{S} = \frac{4\pi \varrho_0 R^3}{3\varepsilon_0}$$

$$\int_0^{\pi} \int_0^{2\pi} \vec{E} \cdot \vec{e}_r \; r^2 \sin\vartheta \, d\varphi \, d\vartheta = \frac{4\pi \varrho_0 R^3}{3\varepsilon_0}$$

→ z úvahy o symetrii: $\vec{E} = E(r)\vec{e}_r$

$$\Rightarrow \int_0^{\pi} \sin\vartheta \, d\vartheta \int_0^{2\pi} d\varphi \left(E(r) r^2 \right) = \frac{4\pi \varrho_0 R^3}{3\varepsilon_0} =$$

$$= 4\pi E(r) r^2 = \frac{4\pi \varrho_0 R^3}{3\varepsilon_0} \quad \Rightarrow \quad E(r) = \frac{\varrho_0 R^3}{3\varepsilon_0 r^2} \qquad r > R$$

pro $r < R$: $\quad 4\pi r^2 E(r) = \frac{4}{3} \frac{\pi \varrho_0}{\varepsilon_0} r^3 \quad \Rightarrow \quad E(r) = \frac{\varrho_0 r}{3\varepsilon_0}$

$$\nabla^2 \varphi(r) = -\frac{\varrho}{\varepsilon}$$

$$\frac{1}{r}(r\varphi)'' = -\frac{\varrho_0}{\varepsilon} \quad \Rightarrow \quad (r\varphi)'' = -\frac{\varrho_0}{\varepsilon} r \quad / \int$$

$$(r\varphi)' = -\frac{1}{2}\frac{\varrho_0}{\varepsilon} r^2 + A$$

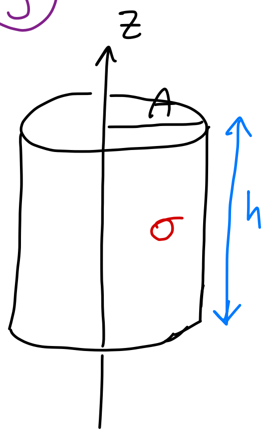
$$r\varphi = -\frac{1}{6}\frac{\varrho_0}{\varepsilon} r^3 + Ar + C$$

$$\varphi = -\frac{1}{6}\frac{\varrho_0}{\varepsilon} r^2 + A + \frac{C}{r}$$

$r > r_0 \quad \varphi = A + \frac{C}{r}$

$r < r_0 \quad \varphi = -\frac{1}{6}\frac{\varrho_0}{\varepsilon} r^2 + A + \frac{C}{r}$

5



$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow \oint_S \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$$

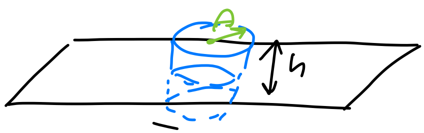
$$\cancel{h}\cdot \cancel{2\pi} r\, E(r) = \sigma\, \cancel{2\pi} A \cdot \cancel{h}$$

$$E(r) = \frac{\sigma A}{r} \qquad r>A$$

$$\Rightarrow \phi = \int \frac{\sigma A}{r} dr = \sigma A \ln|r| + C$$

$$\rightarrow \phi = \sigma A \ln\left(\frac{r}{A}\right)$$

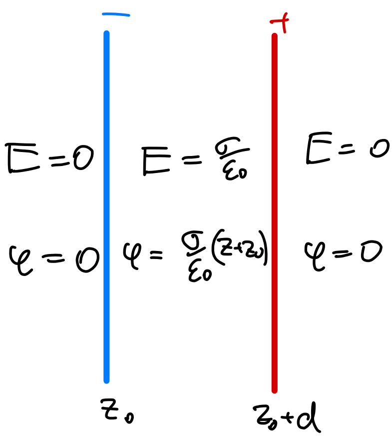
6



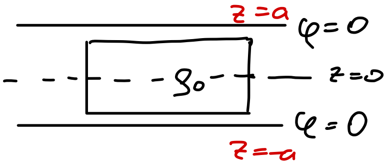
$$\Theta(z-z_0) = \begin{cases} 0 & z<z_0 \\ 1 & z>z_0 \end{cases}$$

$$\oint \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0} = \frac{\sigma\cdot\pi A^2}{\varepsilon_0}$$

$$2E\, \cancel{\pi A^2} = \frac{\sigma\, \cancel{\pi A^2}}{\varepsilon_0} \Rightarrow E(z) = \frac{\sigma}{2\varepsilon_0} \rightarrow \varphi = \frac{\sigma}{2\varepsilon_0} z$$



7



$$\nabla^2 \varphi = -\frac{\rho}{\varepsilon} = -\frac{\rho_0}{\varepsilon}$$

$$\frac{d^2\varphi}{dz^2} = -\frac{\rho_0}{\varepsilon} \Rightarrow \varphi(z) = -\frac{\rho_0}{\varepsilon}\frac{z^2}{2} + Az + B$$

$$\varphi(z) = -\frac{\rho_0}{2\varepsilon}(z-a)(z+a) =$$

$$= -\frac{\rho_0}{2\varepsilon_0}(z^2 - a^2)$$

8

$$\phi(r) = \frac{q}{4\pi\varepsilon_0 a}\left(1 + \frac{a}{r}\right)e^{-\frac{2r}{a}}$$

$$\vec{E} = -\nabla\phi(r) = -\phi'(r)\,\vec{e}_r = \frac{-q}{a4\pi\varepsilon_0}\left[\frac{a}{r^2}e^{-\frac{2r}{a}} + \left(1+\frac{a}{r}\right)\left(-\frac{2}{a}\right)e^{-\frac{2r}{a}}\right]\vec{e}_r =$$

$$= \frac{q}{a4\pi\varepsilon_0}e^{-\frac{2r}{a}}\left[\frac{a}{r^2} + \frac{2}{a} + \frac{2}{r}\right]\vec{e}_r$$

$$\oint_S \vec{E}\cdot d\vec{S} = \int \vec{E}\, r^2\, d\Omega = \frac{q}{a\varepsilon_0}e^{-\frac{2r}{a}}\left(a + \frac{2}{a}r^2 + 2r\right) = \frac{Q(r)}{\varepsilon_0}$$

$$Q(r) = \frac{q}{a}e^{-\frac{2r}{a}}\left(a + \frac{2}{a}r^2 + 2r\right) \qquad \lim_{r\to\infty} Q(r) = 0$$

$$\lim_{r\to 0} Q(r) = q$$

9

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0} = \frac{1}{r}(r\phi)''$$

$$r\phi = \frac{q}{4\pi\varepsilon_0 a}(r+a)e^{-\frac{2r}{a}}$$

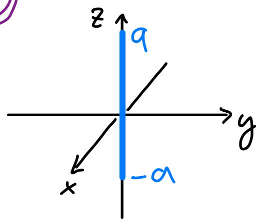
$$(r\phi)' = \frac{q}{4\pi\varepsilon_0 a}e^{-\frac{2r}{a}}\left(1 + (r+a)\left(-\frac{2}{a}\right)\right) = \frac{q}{4\pi\varepsilon_0 a}e^{-\frac{2r}{a}}\left(-1 - \frac{2r}{a}\right) =$$

$$= \frac{-q}{4\pi\varepsilon_0 a}e^{-\frac{2r}{a}}\left(\frac{2r}{a} + 1\right)$$

$$(r\phi)'' = \frac{-q}{4\pi\varepsilon_0 a}e^{-\frac{2r}{a}}\left[\cancel{\frac{2}{a}} + \left(\frac{2r}{a}\cancel{+1}\right)\left(-\frac{2}{a}\right)\right] = \frac{q}{4\pi\varepsilon_0 a}\frac{4r}{a^2}e^{-\frac{2r}{a}}$$

$$\nabla^2\phi = \frac{1}{r}(r\phi)'' = \frac{q}{\pi\varepsilon_0 a^3}e^{-\frac{2r}{a}} = -\frac{\rho}{\varepsilon_0} \Rightarrow \rho = -\frac{q}{a^3\pi}e^{-\frac{2r}{a}}$$

10



$$\Delta_x G(\vec{x}, \vec{x}') = \delta(\vec{x}, \vec{x}')$$

$$\phi = \int G(\vec{x}, \vec{x}')\, \rho(\vec{x}')\, d^3\vec{x}'$$

$$G = \frac{1}{4\pi}\frac{1}{|\vec{x} - \vec{x}'|} \qquad \rho(\vec{x}') = \lambda\,\delta(x')\delta(y')$$

$$\phi = \frac{1}{4\pi}\int \frac{1}{|\vec{x}-\vec{x}'|}\lambda\delta(x')\delta(y')\,dx'dy'dz' =$$

$$\phi = \frac{\lambda}{4\pi}\int_{-a}^{a}\frac{1}{\sqrt{x^2+y^2+(z-z')^2}}dz' = \frac{\lambda}{4\pi}\int_{-a}^{a}\frac{1}{\sqrt{x^2+y^2}}\frac{1}{\sqrt{1+\frac{(z-z')^2}{x^2+y^2}}}dz' =$$

$$= \frac{-\lambda}{4\pi}\left[\operatorname{argsinh}\left(\frac{z-z'}{\sqrt{x^2+y^2}}\right)\right]_{-a}^{a} = \frac{-\lambda}{4\pi}\left[\operatorname{argsinh}\left(\frac{z-a}{\sqrt{x^2+y^2}}\right) - \operatorname{argsinh}\left(\frac{z+a}{\sqrt{x^2+y^2}}\right)\right]$$

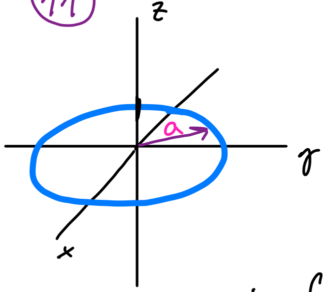
$$= \frac{\lambda}{4\pi}\log\left(\frac{z+a+\sqrt{(z+a)^2+x^2+y^2}}{z-a+\sqrt{(z-a)^2+x^2+y^2}}\right)$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$x = \frac{e^{\operatorname{ash} x} - e^{-\operatorname{ash} x}}{2} \Rightarrow e^{2\operatorname{ash} x} - 2x e^{\operatorname{ash} x} - 1 = 0$$

$$\Rightarrow e^{\operatorname{ash} x} = x + \sqrt{x^2+1} \Rightarrow \operatorname{ash} x = \log\left(x + \sqrt{x^2+1}\right)$$

11



$$-\Delta_x \, G(\vec{x}, \vec{x}') = \delta^3(\vec{x} - \vec{x}')$$

$$\rho(\vec{x}) = \lambda \, \delta(z) \, \delta\left(\sqrt{x^2+y^2} - a\right) = \lambda \delta(z) \delta(R-a)$$

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi} \frac{1}{|\vec{x} - \vec{x}'|}$$

$$\phi = \frac{\lambda}{4\pi} \int \frac{1}{|\vec{x} - \vec{x}'|} \delta(R-a) \, R \, dR \, d\varphi =$$

$$= \frac{\lambda}{4\pi} \int_0^{2\pi} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} \delta(R-a) \, R \, dR \, d\varphi$$

$$= \frac{\lambda}{4\pi} \int_0^{2\pi} \frac{1}{\sqrt{(x - a\cos\varphi')^2 + (y - a\sin\varphi')^2 + z^2}} d\varphi' =$$

$$= \frac{\lambda}{4\pi} \int_0^{2\pi} \frac{1}{\sqrt{x^2+y^2+z^2+a^2 - 2ax\cos\varphi' - 2ay\sin\varphi'}} d\varphi' =$$

$$= \frac{\lambda}{4\pi} \int_0^{2\pi} \frac{1}{\sqrt{R^2+a^2+z^2 - 2aR(\cos\varphi\cos\varphi' + \sin\varphi'\sin\varphi)}} d\varphi' =$$

$$= \frac{\lambda}{4\pi} \int_0^{2\pi} \frac{1}{\sqrt{R^2 - 2aR\cos(\varphi - \varphi') + a^2 + z^2}} d\varphi' = \left| 2b = \varphi - \varphi' \right|$$

$$= \frac{\lambda}{4\pi} \int_0^{\pi} \frac{2\,db}{\sqrt{R^2 - 4aR\cos^2 b + 2aR + a^2 + z^2}} =$$

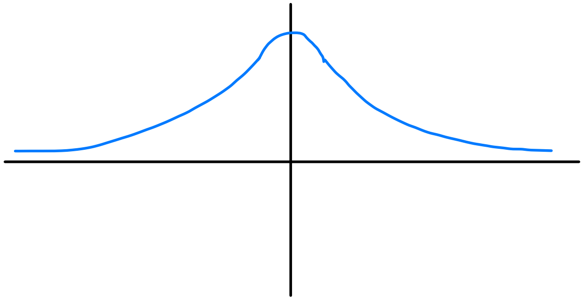
$$k^2 = \frac{4aR}{(R+a)^2 + z^2}$$

$$= \frac{\lambda}{2\pi} \int_0^{\pi} \frac{1}{\sqrt{(R+a)^2 + z^2 - 4aR\cos^2 b}} db$$

$$= \frac{\lambda}{2\pi} \frac{1}{\sqrt{(R+a)^2 + z^2}} \int_0^{\pi} \frac{db}{\sqrt{1 - k^2\cos^2 b}}$$

$\to$ na ose z $(x, y = 0)$:

$$\phi(z) = \frac{\lambda}{4\pi}\int_0^{2\pi}\frac{1}{\sqrt{z^2+a^2}}\,d\varphi' = \frac{\lambda}{2\sqrt{z^2+a^2}}$$



(12) $\nabla(2z^2 - x^2 - y^2) = (-2x, -2y, 4z) = -2x\vec{e}_x - 2y\vec{e}_y + 4z\vec{e}_z = A$

$\nabla 2r^2 P_2(\cos\vartheta) = \nabla 2r^2 \frac{1}{2}(3\cos^2\vartheta - 1) = \nabla r^2(3\cos^2\vartheta - 1) = B$

$$\nabla = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\vartheta}\vec{e}_\vartheta + \cancelto{0}{\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}}\vec{e}_\varphi$$

$$P_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}\frac{d}{dx}\left((x^2-1)x\right) =$$

$$= \frac{1}{2}\left(x^2 - 1 + 2x^2\right) = \frac{1}{2}(3x^2 - 1)$$

$$\nabla r^2(3\cos^2\vartheta - 1) = 2r(3\cos^2\vartheta - 1)\vec{e}_r - 6\overbrace{r\cos\vartheta}^{z}\sin\vartheta\,\vec{e}_\vartheta$$

$$\vec{e}_r = \frac{\partial x}{\partial r}\vec{e}_x + \frac{\partial y}{\partial r}\vec{e}_y + \frac{\partial z}{\partial r}\vec{e}_z = \cos\varphi\sin\vartheta\,\vec{e}_x + \sin\varphi\sin\vartheta\,\vec{e}_y + \cos\vartheta\,\vec{e}_z$$

$$\vec{e}_\vartheta = \cos\varphi\cos\vartheta\,\vec{e}_x + \sin\varphi\cos\vartheta\,\vec{e}_y - \sin\vartheta\,\vec{e}_z$$

$$\nabla(\cdots) = 2(3\cos^2\vartheta - 1)(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z)$$
$$-6\left(\cancel{\cos^2\vartheta}\, x\vec{e}_x + \cancel{\cos^2\vartheta}\, y\vec{e}_y - \sin^2\vartheta\, z\vec{e}_z\right) =$$
$$= -2x\vec{e}_x - 2y\vec{e}_y + 6\cos^2\vartheta\, z\vec{e}_z + 6\sin^2\vartheta\, z\vec{e}_z - 2z\vec{e}_z =$$
$$= -2x\vec{e}_x - 2y\vec{e}_y + 4z\vec{e}_z$$

$$\nabla \cdot A = -2 - 2 + 4 = 0 \qquad 3r \sin 2\vartheta$$

$$\nabla \cdot B = 2r(3\cos^2\vartheta - 1)\vec{e}_r - 6r\cos\vartheta \sin\vartheta \, \vec{e}_\vartheta$$

$$\nabla \cdot B = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(B h_2 h_3) + \frac{\partial}{\partial q_2}(B h_1 h_3) + \frac{\partial}{\partial q_3}(B h_2 h_1)\right]$$

$$= \frac{1}{r^2 \sin\vartheta}\left[\frac{\partial}{\partial r}(B_r r^2 \sin\vartheta) + \frac{\partial}{\partial \vartheta}(B_\vartheta r \sin\vartheta) + \frac{\partial}{\partial \varphi}(B_\varphi r)\right] =$$

$$= \frac{1}{\cdots}\left[\sin 6\cancel{r^2}(3\cos^2\vartheta - 1) - 3\cancel{r^2}(2\cos 2\vartheta \sin\vartheta + \sin 2\vartheta \cos\vartheta)\right]$$

$$= 3\left[6\cos^2\vartheta - 6 - 2\cos 2\vartheta + 2\cos\vartheta\cos\vartheta\right] =$$

$$= 3\left[6\cos 2\vartheta - 6 - \cancel{2\cos^2\vartheta} + 2\sin^2\vartheta + \cancel{2\cos^2}\vartheta\right]$$

$$\Delta P_3(x,y,z) = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0$$

$$P_3(x,y,z) = x^3 + x^2(Ay + Bz) + x(Cy^2 + Dy + Ez^2 + Fz)$$

$$\left.\begin{aligned} \hookrightarrow \quad & 6x + 2(Ay + Bz) + \\ + \quad & x2C \\ + \quad & x2E \end{aligned}\right\} \begin{aligned} & 2C + 2E = 6 \Rightarrow C = 3 - E \\ & A = B = 0 \end{aligned}$$

$$P_3(x,y,z) = x^3 + x(3y^2 + Dy + E(z^2 - y^2) + Fz)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\partial_i u = \varepsilon_{ij} \cdot \partial_j v$$

$$\partial_i u = \varepsilon_{ijk} \partial_j v \, \partial_k w$$

$$\nabla f = \frac{1}{h_i}\frac{\partial f}{\partial q_i}\vec{e}_i$$

$$\Delta A \quad ; \quad f = R(r)\,\Theta(\vartheta)\,\phi(\varphi)$$

$$\nabla \cdot \nabla f = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial q_1}\left(\frac{\partial f}{\partial q_1}\frac{h_2 h_3}{h_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{\partial f}{\partial q_2}\frac{h_1 h_3}{h_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{\partial f}{\partial q_3}\frac{h_1 h_2}{h_3}\right)\right)$$

$$\nabla^2 f = \frac{1}{r^2 \sin\vartheta}\left(\underline{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r} r^2 \sin\vartheta\right)} + \underline{\frac{\partial}{\partial \vartheta}\left(\frac{\partial f}{\partial \vartheta}\sin\vartheta\right)} + \underline{\frac{\partial}{\partial \varphi}\left(\frac{\partial f}{\partial \varphi}\frac{1}{\sin\vartheta}\right)}\right)$$

$$= \frac{1}{r^2 \sin\vartheta}\left(\underline{\sin\vartheta\,\Theta\,\Phi\frac{\partial}{\partial r}\left(R' r^2\right)} + \underline{R\,\Phi\frac{\partial}{\partial \vartheta}\left(\Theta'\sin\vartheta\right)} + \underline{\frac{1}{\sin\vartheta}R\,\Theta\,\Phi''}\right) = 0$$

$$\Rightarrow \quad \frac{\frac{d}{dr}\left(R' r^2\right)}{\cancel{r^2}R} + \frac{\frac{d}{d\vartheta}\left(\Theta'\sin\vartheta\right)}{\cancel{r^2}\sin\vartheta\,\Theta} + \frac{\Phi''}{\cancel{r^2}\sin^2\vartheta\,\Phi} = 0$$

$$\frac{(R' r^2)'}{R} = \lambda = -\frac{(\Theta'\sin\vartheta)'}{\sin\vartheta\,\Theta} - \frac{\Phi''}{\sin^2\vartheta\,\Phi}$$

$$\Rightarrow \quad (R' r^2)' - \lambda R = 0$$

$$\lambda \sin^2\vartheta + \frac{(\Theta'\sin\vartheta)'}{\Theta} = \frac{\Phi''}{\Phi} = -m^2$$

$$\Rightarrow \quad \Phi'' + m^2\Phi = 0$$

$$(\Theta'\sin\vartheta)' + (\lambda\sin^2\vartheta + m^2)\,\Theta = 0$$

$$\nabla^2 f = 0, \quad f = R(r)\,\Theta(\vartheta)\,\Phi(\varphi)$$

1) $(R' r^2)' - \lambda R = 0$

2) $\Phi'' + m^2 \Phi = 0$

3) $(\Theta' \sin\vartheta)' + (\lambda \sin^2\vartheta + m^2)\,\Theta = 0$

2) $\Phi = A\cos(m(\varphi - \varphi_0)) \sim A e^{im\varphi}$

$\Phi(\varphi) = \Phi(\varphi + 2n\pi) \Rightarrow m \in \mathbb{Z}$

1) $R'' r^2 + 2R' r - \lambda R = 0$

$R = r^\alpha \Rightarrow \alpha(\alpha - 1) + 2\alpha - \lambda = 0$

$\alpha^2 + \alpha - \lambda = 0$

$$\alpha = \frac{-1 \pm \sqrt{1 + 4\lambda}}{2}$$

3) substituce $x = \cos\vartheta$

$dx = -\sin\vartheta\, d\vartheta$

$$\left(\sin\vartheta \frac{d\Theta}{d\vartheta}\right)' = \left(\sin\vartheta \frac{dx}{d\vartheta} \frac{d\Theta}{dx}\right)' = \left(\overbrace{-\sin^2\vartheta}^{x^2-1} \frac{d\Theta}{dx}\right)$$

$$\frac{d}{dx}\left((1 - x^2)\frac{d}{dx}P(x)\right) - \left(\lambda - \frac{m^2}{1 - x^2}\right)P(x) = 0$$

$\Theta(\vartheta) = \Theta(\vartheta + 2n\pi) \Rightarrow \lambda = \ell(\ell + 1)$

$\Theta(\vartheta) = P_\ell^m(\cos\vartheta) \qquad \Rightarrow R = r^\ell, r^{-\ell-1}$

$$Y_\ell^m(\vartheta,\varphi) = N P_\ell^m(\cos\vartheta)\, e^{im\varphi}$$, N je normalizace

$$f(r,\vartheta,\Phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{m,\ell}\, r^\ell Y_\ell^m(\vartheta,\varphi) + \hat{C}_{m,\ell}\, r^{-\ell-1} Y_\ell^m(\vartheta,\varphi)$$

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_\ell^m(x)\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_\ell^m(x) = 0$$

Veta (Sturm-Liouvillova)

Pre differenciálnu rovnicu v tvare

$$\frac{d}{dx}\Big[\underbrace{p(x)}_{\geq 0}\frac{d}{dx}y(x)\Big] + q(x)y(x) = -\lambda \underbrace{w(x)}_{\geq 0} y(x) \qquad \text{pro} \quad x\in[a,b]$$

podm: $\alpha_1 y(a) + \alpha_2 y'(a) = 0 \qquad \alpha_1^2 + \alpha_2^2 > 0$

$\beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \beta_1^2 + \beta_2^2 > 0$

váhová míra

existuje ON báze $\mathcal{L}^2([a,b], w(x))$, jako báze vlastních vektorů operátoru $\frac{d}{dx}\left[p(x)\frac{d}{dx}\,\bullet\,\right] + q(x)\,\bullet$ s vlastními čísly λ_i.

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_\ell^m(x)\right] + \Big[\underbrace{\ell(\ell+1)}_{\lambda} - \frac{m^2}{1-x^2}\Big]P_\ell^m(x) = 0$$

Protáhlé elipsoidální súradnice:

$$x = \sqrt{s^2-a^2}\,\sin\vartheta\,\cos\varphi \qquad ; \quad s > a$$
$$y = \sqrt{s^2-a^2}\,\sin\vartheta\,\sin\varphi$$
$$z = s\cos\vartheta$$

$$\Delta\phi = 0 \qquad ; \quad \phi(s)$$

$$\frac{\partial\vec{x}}{\partial s} = \frac{s}{\sqrt{s^2-a^2}}\sin\vartheta\cos\varphi\,\vec{e}_x + \frac{s}{\sqrt{s^2-a^2}}\sin\vartheta\sin\varphi\,\vec{e}_y + \cos\vartheta\,\vec{e}_z$$

$$\frac{\partial\vec{x}}{\partial\vartheta} = \sqrt{s^2-a^2}\cos\vartheta\cos\varphi\,\vec{e}_x + \sqrt{s^2-a^2}\cos\vartheta\sin\varphi\,\vec{e}_y - s\sin\vartheta\,\vec{e}_z$$

$$\frac{\partial\vec{x}}{\partial\varphi} = \sqrt{s^2-a^2}\sin\vartheta(-\sin\varphi)\,\vec{e}_x + \sqrt{s^2-a^2}\sin\vartheta\cos\varphi\,\vec{e}_y$$

$$h_s^2 = \frac{s^2\sin^2\vartheta + \cos^2\vartheta}{s^2-a^2} = \frac{s^2-a^2\cos^2\vartheta}{s^2-a^2}$$

$$h_\vartheta^2 = (s^2-a^2)\cos^2\vartheta + s^2\sin^2\vartheta = s^2-a^2\cos^2\vartheta$$

$$h_\varphi^2 = (s^2-a^2)\sin^2\vartheta$$

$$\Rightarrow h_s = \sqrt{\frac{s^2-a^2\cos^2\vartheta}{s^2-a^2}} \qquad h_\vartheta = \sqrt{s^2-a^2\cos^2\vartheta} \qquad h_\varphi = \sqrt{s^2-a^2}\sin\vartheta$$

$$\nabla^2\phi(s) = \frac{1}{h_s\cancel{h_\vartheta}h_\varphi}\left(\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial s}\,\frac{h_\vartheta h_\varphi}{h_s}\right)\right) = 0$$

$$\frac{h_\vartheta h_\varphi}{h_s} = (s^2-a^2)\sin\vartheta$$

$$\Rightarrow \left[(s^2-a^2)\,\phi'\right]' = 0 \quad \Rightarrow \quad \phi'(s) = \frac{A}{s^2-a^2} = A\left(\frac{1}{s+a} - \frac{1}{s-a}\right)$$

$$\Rightarrow \phi(s) = A \operatorname{arg\,tgh}\left(\frac{s}{a}\right) + B \qquad \frac{s}{a} \in (-1,1)$$ → nevyhovuje

$$\phi(s) = A \operatorname{argcotgh}\left(\frac{s}{a}\right) + B \qquad \frac{s}{a} \in \mathbb{R} \setminus (-1,1)$$

$$\operatorname{tgh}(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \leftrightarrow \quad x = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y}-1}{e^{2y}+1}$$

$$x\left(e^{2y}+1\right) = e^{2y} - 1$$

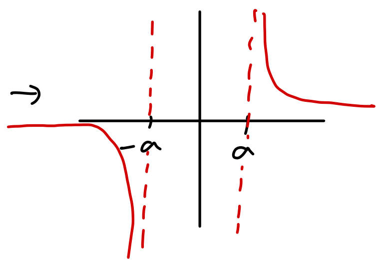
$$e^{2y}(1-x) = x+1 \quad \Rightarrow \quad y = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$$

$$\Rightarrow \phi(s) = A\left(\ln\left(\frac{a+s}{a-s}\right) + B\right) \longrightarrow \operatorname{arctgh}$$

$$\phi(s) = A\left(\ln\left(\frac{s+a}{s-a}\right) + B\right) \to \operatorname{arccotgh}$$

$$\lim_{s\to\infty} \phi(s) = 0 \Rightarrow B = 0$$

$$\phi(s) = A \ln\left(\frac{s+a}{s-a}\right) = A \operatorname{argcotanh}\left(\frac{s}{a}\right) \to$$



$$\operatorname{acth}(x) = \int \frac{1}{1-x^2}\,dx = \int \frac{1}{x^2} \frac{1}{1-\frac{1}{x^2}} =$$

$$\approx \int \frac{1}{x^2} = -\frac{1}{x}$$

$$\cancel{\Rightarrow \phi(s) \overset{\text{velké } s}{\approx} -\frac{Aa}{s} = \frac{Q}{4\pi\varepsilon_0 a} \overset{s=a}{\Rightarrow} A = -\frac{Q}{4\pi\varepsilon_0 a}}$$

$$\cancel{\Rightarrow \phi(s) = \frac{Q}{4\pi\varepsilon_0 a} \ln\left(\frac{s-a}{s+a}\right)}$$

$$s^2\cos^2\vartheta + a^2 - 2as\cos\vartheta$$

$$z-a+\sqrt{R^2+(z-a)^2} = s\cos\vartheta - a + \sqrt{(s^2-a^2)\sin^2\vartheta + (s\cos\vartheta-a)^2} =$$

$$= s\cos\vartheta - a + \sqrt{s^2 + a^2\cos^2\vartheta - 2as\cos\vartheta} =$$

$$= s\cos\vartheta - a + \sqrt{(s-a\cos\vartheta)^2} = s\cos\vartheta - a + s - a\cos\vartheta =$$
$$= (s-a)(1+\cos\vartheta)$$

$$z+a+\sqrt{R^2+(z-a)^2} = \dots = (s+a)(1+\cos\vartheta)$$

$$\Rightarrow \ln\left(\frac{z-a+\sqrt{R^2+(z-a)^2}}{z+a+\sqrt{R^2+(z+a)^2}}\right) = \ln\left(\frac{(s-a)\cancel{(1+\cos\vartheta)}}{(s+a)\cancel{(1+\cos\vartheta)}}\right) = \ln\left(\frac{s-a}{s+a}\right)$$

$$\Phi = A\ln\frac{s+a}{s-a}$$

$$Q = \frac{1}{\varepsilon_0}\oint \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\oint \vec{E}\cdot\vec{e}_s\, h_\vartheta h_\varphi\, d\vartheta\, d\varphi$$

$$\vec{E} = -\nabla\phi = -\nabla\phi = -\frac{1}{h_s}\partial_s \Phi\, \vec{e}_s$$

$$\Rightarrow \quad Q = \varepsilon_0 \oint -\frac{h_\vartheta h_\varphi}{h_s}\partial_s \Phi\, d\vartheta\, d\varphi = -$$

$$= -A\varepsilon_0 \oint (s^2-a^2)\sin\vartheta\left(\frac{1}{s+a} - \frac{1}{s-a}\right) d\vartheta\, d\varphi =$$

$$= 2Aa\varepsilon_0 \oint \sin\vartheta\, d\vartheta\, d\varphi = \frac{2Aa}{\varepsilon_0}\, 2\pi \int_0^\pi \sin\vartheta\, d\vartheta =$$

$$= 8\pi A a \varepsilon_0 = Q \quad \Rightarrow$$

$$A = \frac{Q}{8\pi a\varepsilon_0}$$

$$\Rightarrow \phi = \frac{Q}{8\pi a \varepsilon_0}\ln\left(\frac{s-a}{s+a}\right)$$

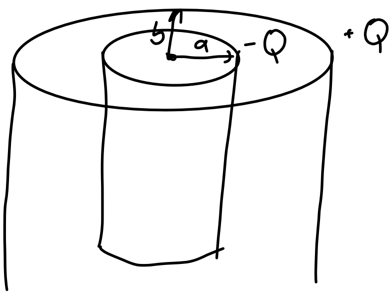
$$\phi = \ln \frac{z+a+\sqrt{R^2+(z+a)^2}}{z-a+\sqrt{R^2+(z-a)^2}} \sim \ln \frac{z+a+a}{z-a+a} = \ln \frac{z+2a}{z} \to \infty$$

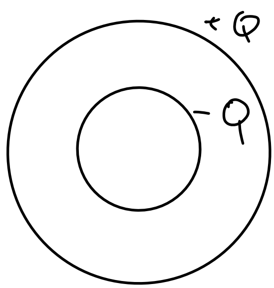
$$\approx \ln \frac{z+a+a\left(1+\frac{z}{a}+\frac{R^2}{2a^2}\right)}{z-a+a\left(1-\frac{z}{a}+\frac{R^2}{2a^2}\right)} =$$

$$= \ln \frac{z+2a+z+\frac{R^2}{2a}}{\cancel{z}-\cancel{z}+\frac{R^2}{2a}} = \ln \frac{2z+2a+\frac{R^2}{2a^2}}{\frac{R^2}{2a^2}} =$$

$$= \ln \frac{2a^2}{R^2}\left(2z+2a+\frac{R^2}{2a^2}\right)$$

$$\phi = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r} \to U = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R} = \frac{Q}{C} \Rightarrow C = 4\pi\varepsilon_0 R$$





$$\varepsilon_0 \oint \vec{E}\cdot d\vec{S} = Q$$

$$\parallel$$

$$\varepsilon_0 \int E(R)\, R\, d\varphi\, dz = Q$$

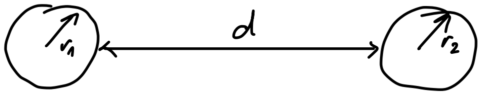
$$\Rightarrow E(R) = \frac{Q}{2\pi\varepsilon_0 R}$$

$$a < R < b$$

$$\Rightarrow U = -\int_a^b \frac{Q}{2\pi\varepsilon_0 R}\, dR = -\frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) = \frac{Q}{2\pi\varepsilon_0}\ln\left(\frac{a}{b}\right)$$

$$\Rightarrow C = \frac{2\pi\varepsilon_0}{\ln\left(\frac{a}{b}\right)}$$

$$Q_a = \sum C_{ab} U_b$$



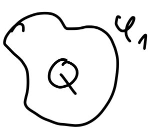
$$\varphi_a = \frac{1}{4\pi\varepsilon_0}\frac{Q_a}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{Q_b}{d}$$

$$\varphi_a = \frac{1}{4\pi\varepsilon_0}\frac{Q_a}{d} + \frac{1}{4\pi\varepsilon_0}\frac{Q_b}{r_2}$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \frac{1}{4\pi\varepsilon_0}\underbrace{\begin{pmatrix}\frac{1}{r_1} & \frac{1}{d}\\ \frac{1}{d} & \frac{1}{r_2}\end{pmatrix}}_{C^{-1}}\begin{pmatrix}Q_a\\ Q_b\end{pmatrix}$$

$$\frac{1}{r_1 r_2} - \frac{1}{d^2} = \frac{d^2 - r_1 r_2}{r_1 r_2 d^2}$$

$$\Rightarrow C = 4\pi\varepsilon_0 \frac{r_1 r_2 d^2}{d^2 - r_1 r_2}\begin{pmatrix}\frac{1}{r_2} & -\frac{1}{d}\\ -\frac{1}{d} & \frac{1}{r_1}\end{pmatrix} = 4\pi\varepsilon_0\begin{pmatrix} r_1 & -\frac{r_1 r_2}{d}\\ -\frac{r_1 r_2}{d} & r_2\end{pmatrix}$$

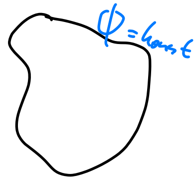


C známe

$$\begin{pmatrix}Q\\ -Q\end{pmatrix} = \begin{pmatrix}C_{11} & C_{12}\\ C_{12} & C_{22}\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \rightarrow \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \frac{1}{C_{11}C_{22} - C_{12}^2}\begin{pmatrix}C_{22} & -C_{12}\\ -C_{12} & C_{11}\end{pmatrix}\begin{pmatrix}Q\\ -Q\end{pmatrix}$$

$$U = (\varphi_1 - \varphi_2) = \frac{Q}{C_{11}C_{22} - C_{12}^2}\left(C_{22} + C_{12} + C_{12} + C_{11}\right)$$

$$\Rightarrow C = \frac{C_{11}C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_1C_2}$$



$\xrightarrow{\phi \to 0}$

$$\phi(\partial V) = U$$
$$\phi(\infty) = 0$$

$$W[\phi] = \frac{\varepsilon_0}{2}\int (\nabla\varphi)^2 \, dV = \frac{\varepsilon_0}{2}\int \left(\frac{\partial}{\partial x_i}\varphi\right)^2 dV \geq \frac{1}{2}CU^2$$

$$= \frac{\varepsilon_0}{2}\int (\partial_i \varphi)^2$$

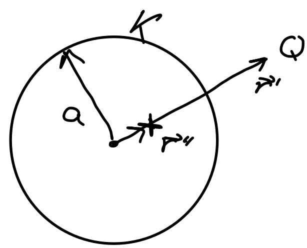
$$W[\phi+\delta] = \frac{\varepsilon_0}{2}\int (\nabla\varphi)^2 + (\nabla\delta)^2 + 2\nabla\varphi\cdot\nabla\delta =$$

Greenova fce bod. náboje v blízkosti vodivé koule

$$\Delta\phi(\vec r) = -\frac{Q}{\varepsilon_0}\delta(\vec r - \vec r') \qquad \Omega = \mathbb{R}^3 \setminus K_a(0)$$
$$\phi(\infty) = 0$$
$$\phi(\partial K_a) = U$$

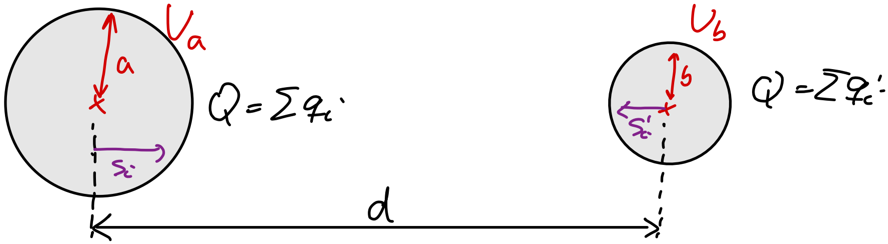
$$\phi(\vec r) = \frac{Ua}{|\vec r|} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec r - \vec r'|} - \frac{a}{|\vec r'|}\frac{1}{|\vec r - \vec r''|}\right)$$

$$\vec r'' = \frac{a^2}{|\vec r'|^2}\vec r'$$



$$\phi(\partial K_a) = U + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec r - \vec r'|} - \frac{q}{|\vec r'|}\frac{1}{\left|\vec r - \frac{a^2}{|\vec r'|^2}\vec r'\right|}\right) =$$

$$= U + \frac{Q}{4\pi\varepsilon_0}\left(\cancel{\frac{1}{|\vec r - \vec r'|}} - \frac{\cancel{q}}{\cancel{|\vec r'|}}\cancel{\frac{1}{\left|\frac{q}{\cancel{|\vec r'|}}\vec r - \frac{a}{\cancel{|\vec r'|}}\vec r'\right|}}\right) = U$$



$$\phi = \sum_{i=0}^{\infty} \frac{q_i}{|\vec{r} - s_i \vec{e}|} + \frac{q_i'}{|\vec{r} - (d - s_i')\vec{e}|} = \frac{q_0}{|\vec{r}|} + \frac{q_0'}{|\vec{r} - d\vec{e}|} + \frac{q_1}{|\vec{r} - s_1 \vec{e}|} + \dots$$

$$\phi = \frac{U_a a}{|\vec{r}|} + \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r} - \vec{r}'|} - \frac{a}{|\vec{r}''|}\,\frac{1}{|\vec{r} - \vec{r}'''|}\right) \qquad \vec{r}'' = \frac{a^2}{|\vec{r}'|^2}\vec{r}'$$

$q_0 = aU_a$ $\qquad$ $q_0' = bU_b$

$s_0 = 0$ $\qquad$ $s_0' = 0$

$q_1 = -\frac{a}{d} q_0'$ $\qquad$ $q_1' = -\frac{b}{d} q_0$

$s_1 = \frac{a^2}{d}$ $\qquad$ $s_1' = \frac{b^2}{d}$

$$q_{i+1} = -q_i' \frac{a}{d - s_i'} \qquad s_{i+1} = \frac{a^2}{d - s_i'}$$

$$Q_a = aU_a - \frac{ab}{d}U_b + \frac{a^2 b}{d^2}U_a - \frac{a^2 b^2}{d^3\left(1 - \frac{b^2}{a^2}\right)\left(1 - \frac{a^2}{a^2}\right)}U_b$$

$$Q_b = bU_b - \frac{ba}{d}U_a + \frac{b^2 a}{d^2 - a}U_b$$

$$\Downarrow$$

$$\vec{Q} = \begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \begin{pmatrix} a + \frac{a^2 b}{d^2} & -\frac{ab}{d} - \frac{a^2 b^2}{d^3\left(1 - \frac{b^2}{a^2}\right)\left(1 - \frac{a^2}{a^2}\right)} \\ -\frac{ba}{d} & b + \frac{b^2 a}{d^2 - a} \end{pmatrix} \begin{pmatrix} U_a \\ U_b \end{pmatrix}$$

$q_i = \frac{a}{d - s_i'} q_{i-1}'$ $\qquad$ $q_i' = \frac{b}{d - s_i} q_{i-1}$

$s_i = \frac{a^2}{d - s_i'}$ $\qquad$ $s_i' = \frac{b^2}{d - s_{i-1}'}$

$$f(x)\quad \frac{1}{\sqrt{1-2wx+x^2}} = \sum_{\ell=0}^{\infty} x^{\ell} P_{\ell}(w)$$

$$(1+\epsilon)^{-\frac{1}{2}}$$

$$(1+\epsilon)^r = 1 + \binom{r}{1}\epsilon + \binom{r}{2}\epsilon^2 + \binom{r}{3}\epsilon^3 + \dots$$

$$(1+\epsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\epsilon \quad + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\epsilon^2 + \frac{1}{\cancel{6}\,2}\left(-\frac{1}{2}\right)\left(-\frac{\cancel{3}\,1}{2}\right)\left(-\frac{5}{2}\right)\epsilon^3 + \dots$$

$$= 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots$$

$$= 1 - \frac{1}{2}(-2wx+x^2) + \frac{3}{8}\underbrace{(-2wx+x^2)^2}_{\cancel{x^4}-4wx^3+4w^2x^2} - \frac{5}{16}\underbrace{(-2wx+x^2)^3}_{\cancel{x^6-6wx\,x^4+3.4w^2x^2x^2}-8w^3x^3} + \dots$$

$$= 1 + wx - \frac{1}{2}x^2 - \frac{3}{2}wx^3 + \frac{3}{2}w^2x^2 + \frac{5}{8}w^3x^3 + \dots$$

$$= 1 + wx + \left(\frac{3}{2}w^2 - \frac{1}{2}\right)x^2 + \left(\frac{5}{8}w^3 - \frac{3}{2}w\right)x^3 + \dots$$

$w = 1$:

$$\frac{1}{\sqrt{1-2x+x^2}} = \frac{1}{\sqrt{(x-1)^2}} = \frac{1}{x-1} = \sum_{\ell=0}^{\infty} x^{\ell} P_{\ell}(w)$$

$$\| $$

$$1 + x + x^2 + x^3 + \dots \Rightarrow P_{\ell}(1) = 1$$

$w = -1$:

$$\frac{1}{\sqrt{1+2x+x^2}} = \frac{1}{1+x} \Rightarrow P_{\ell}(w) = (-1)^{\ell}$$

$$\frac{1}{\sqrt{1-2\cos\theta\frac{a}{r}+\frac{a^2}{r^2}}} = \frac{r}{\sqrt{r^2-2ar\cos\theta+a^2}} =$$

$$= \frac{r}{\sqrt{(r-a\cos\theta)^2-a^2\cos^2\theta+a^2}} = \frac{r}{\sqrt{(r-a\cos\theta)^2+a^2(1-\cos^2\theta)}} =$$

$$= \frac{r}{\sqrt{(r-a\cos\theta)^2+a^2\sin^2\theta}}$$

$$\frac{1}{|\vec{r}-a\vec{e}_z|} = \sum_l \frac{a^l}{r^{l+1}} P_l(\cos\vartheta) \quad \Big\downarrow \nabla^2$$

$$\nabla^2 \frac{1}{\vec{r}-a\vec{e}_z} = 0 = \sum_l a^l \Delta\left(\frac{P_l(\cos\vartheta)}{r^{l+1}}\right)$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}r^{-l-1}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left((-l-1)\,r^{-l} \right) = \frac{1}{r^2}(l+1)\,l\,r^{-l-3}$$

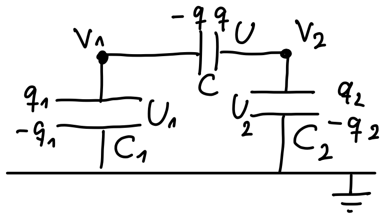
$$= (l+1)(l+2)\,r^{l-1}$$

$$\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}P_l(\cos\vartheta)\right) = \frac{\partial}{\partial\vartheta}\left(-\sin^2\vartheta\, P_l'\right) =$$

$$= -2\sin\vartheta\cos\vartheta\, P_l' + \sin^3\vartheta\, P_l''$$

Př

$\phi(\infty) = 0$



$$\vec{Q} = \overleftrightarrow{C}\,\vec{U}$$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} C_{11} & -C_{12} \\ -C_{12} & C_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$Q_1 = C_{11}\phi_1 - C_{12}\phi_2$$
$$Q_2 = C_{22}\phi_2 - C_{12}\phi_1$$

$$U_1 = \phi_1$$
$$U_2 = \phi_2$$
$$U = \phi_2 - \phi_1 = U_2 - U_1$$
$$Q_1 = q_1 - q$$
$$Q_2 = q_2 + q$$

$$(C_1 + C)U_1 - CU_2 = C_1U_1 - CU = Q_1 = q_1 - q = C_{11}U_1 - C_{12}U_2$$
$$(C_2 + C)U_2 - CU_1 = C_2U_2 + CU = Q_2 = q_2 + q = C_{22}U_2 - C_{12}U_1$$

$$q_1 = C_1U_1$$
$$q_2 = C_2U_2$$
$$q = C\,U$$

$$\Rightarrow C_{12} = C$$
$$C_{11} = C_1 + C$$
$$C_{22} = C_2 + C$$

$$\phi(x) = \frac{1}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}} \frac{1}{r^{l+1}} Y_l^m(\vartheta, \varphi)\, M_l^m$$

$$M_l^m = \int \sqrt{\frac{4\pi}{2l+1}} r'^l\, Y_l^{m*}(\vartheta, \varphi)\, \rho(\vec{x}')\, dV'$$ → sferický multipólový moment

$$Y_l^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\, e^{im\varphi} P_l(\cos\vartheta)$$

Pozn: $\frac{1}{\sqrt{1-2xy+y^2}} = \sum_{n=0}^{\infty} P_n(x)\, y^n$ $\qquad$ $P_0(x)=1$, $P_1(x)=x$

Pr

Axiálne sym. pole na z, daleko

$$\phi(x=y=0,z) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{q_l}{z^{l+1}} \quad ; \quad q_l = \int \rho(\vec{x}')\, r'^{l} P_l(\cos\vartheta')\, dV'$$

$$\phi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, dV'$$

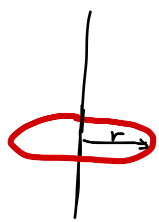
$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\sqrt{x'^2+y'^2+(z-z')^2}} = \frac{1}{\sqrt{r'^2-2zz'+z^2}} =$$

$$= \frac{1}{z}\,\frac{1}{\sqrt{1-\frac{2z'r'}{zr'}+\left(\frac{r'}{z}\right)^2}} = \frac{1}{z}\sum_{l=0}^{\infty} P_l\!\left(\frac{z'}{r'}\right)\cdot\left(\frac{r'}{z}\right)^{l} =$$

$$= \sum_{l=0}^{\infty} P_l(\cos\vartheta')\,\frac{r'^{l}}{z^{l+1}}$$

$$\phi = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{z^{l+1}}\underbrace{\int \rho(\vec{x}')\, P_l(\cos\vartheta')\, r'^{l}\, dV}_{q_l}$$

- pole na ose kružnice

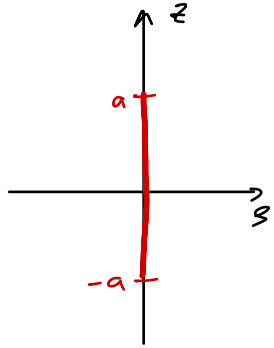


$$\rho(\vec{x}) = \frac{\lambda}{a}\,\delta(r-R)\,\delta\!\left(\vartheta-\frac{\pi}{2}\right)$$

$$\int P_l(\cos\vartheta')\, Q\, \delta(R'-a)\,\delta(\vartheta')\, R'^{l}\, R'^{2}\sin\vartheta'\, d\vartheta'\, d\varphi'\, dR' =$$

$$= 2\pi a \lambda a^{l} P_l(0) \qquad \Rightarrow \phi = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{2\pi Q a^{l+2}}{r^{l+1}}\, P_l(\cos\vartheta)\, P_l(0)$$

0 pre l sudé



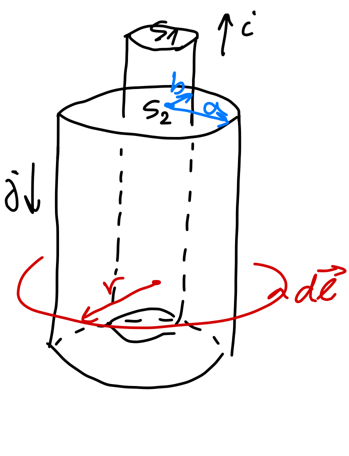
$$\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2a}\ln\frac{z+a+\sqrt{\rho^2+(z+a)^2}}{z-a+\sqrt{\rho^2+(z-a)^2}}$$

$$q_\ell = \int r'^{\ell+1} P_\ell(\cos\vartheta)\,\rho(r')\,dV =$$

$$= \int_{-a}^{a} \lambda\,(\pm z)^{\ell+1} P_\ell(\pm 1)\,dz =$$

$$= \int_{-a}^{a} \lambda\, z^{\ell+1}\,dz = \lambda\left[\frac{z^\ell}{\ell}\right]_{-a}^{a} = \frac{\lambda a^\ell}{\ell}\left[1-(-1)^\ell\right]$$

$$\Rightarrow q_\ell = \begin{cases} 0 & ;\ \ell \text{ sudé} \\ \dfrac{2\lambda a^\ell}{\ell} & ;\ \ell \text{ liché} \end{cases}$$



$$\nabla\times\vec{B} = -\mu_0\vec{j}$$

$$\oint_{\partial S}\vec{B}\cdot d\vec{\ell} = -\mu_0\int_S \vec{j}\cdot d\vec{S}$$

$$\oint_{\partial S}\vec{B}\cdot d\vec{\ell} = -\mu_0\,(i\cdot S_1 - j\cdot S_2)$$

$$2\pi r B = \mu_0 (j\cdot S_2 - i\cdot S_1)$$

$$\Rightarrow B = \frac{\mu_0}{2\pi r}(j\cdot S_2 - i\cdot S_1) \quad r>a$$

$$B = \frac{\mu_0}{2\pi r}(-i\cdot S_1) \quad b<r<a$$

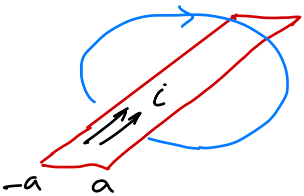
$$B = \frac{\mu_0}{2} \frac{a^2}{r} \vec{e}_\varphi$$

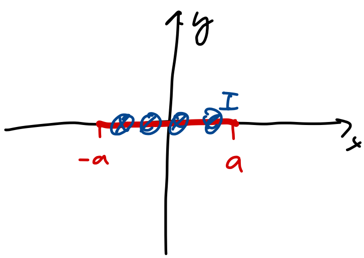
$$\nabla \times = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_1 & h_2 \vec{e}_2 & h_3 \vec{e}_3 \\ \partial q_1 & \partial q_2 & \partial q_3 \\ h_1 \cdot & h_2 \cdot & h_3 \cdot \end{vmatrix} =$$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \vec{e}_1 & h_2 \vec{e}_2 & h_3 \vec{e}_3 \\ \partial q_1 & \partial q_2 & \partial q_3 \\ 0 & 0 & A_z \end{vmatrix} = \frac{\cancel{h_3} \vec{e}_3}{h_1 h_2 \cancel{h_3}} \left(\partial q_1 (h_2 \cdot) - \partial q_2 (h_1 \cdot) \right)$$

$$= \vec{e}_\varphi \frac{\partial A}{\partial r}$$

$$\frac{\mu_0 i}{2} \frac{a^2}{r} = \frac{\partial A}{\partial r} \Rightarrow A(r) = \frac{\mu_0 i}{2} a^2 \ln r + C$$





$$\vec{B} = \frac{\mu_0}{2\pi} \int \vec{j}\,' \times \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} d\vec{r}\,'$$

$$\vec{B} = \frac{\mu_0 I}{4\pi a} \left[\left(\operatorname{arctg} \frac{x-a}{y} - \operatorname{arctg} \frac{x+a}{y} \right) \vec{e}_x + \frac{1}{2} \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2} \vec{e}_y + 0 \vec{e}_z \right]$$

$$[B_x] = \lim_{y \to 0^+} B_x - \lim_{y \to 0^-} B_x = -\frac{\mu_0 I}{4\pi a} 2\pi \vec{e}_x = -\frac{\mu_0 I}{2a} \vec{e}_x \cdot$$

$$\vec{n} \times \vec{e}_x \left(\frac{-\mu_0 I}{2a} \right) = \vec{e}_y \times \vec{e}_x \left(-\frac{\mu_0 I}{2a} \right) = \vec{e}_z \frac{\mu_0 I}{2a}$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} d^3\vec{r}' = \frac{\mu_0}{4\pi}\int_\gamma \frac{I\, d\vec{\ell}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

$$\vec{r} = (r\cos\varphi, r\sin\varphi, z)$$
$$\vec{r}' = (a\cos\varphi', a\sin\varphi', 0)$$

$$\vec{r}-\vec{r}' = (r\cos\varphi - a\cos\varphi', r\sin\varphi - a\sin\varphi', z)$$
$$|\vec{r}-\vec{r}'|^2 = r^2 + a^2 + z^2 - 2ar\cos(\varphi-\varphi')$$

$$d\vec{\ell} = (-a\sin\varphi', a\cos\varphi', 0)$$

$$d\vec{\ell}\times(\vec{r}-\vec{r}') = (a\cos\varphi' z, a\sin\varphi' z, -a\sin\varphi'(r\sin\varphi - a\sin\varphi') - a\cos\varphi'(r\cos\varphi - a\cos\varphi'))$$

$$= (az\cos\varphi', az\sin\varphi', a^2 - ar\cos(\varphi-\varphi'))$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{az\cos\varphi'\vec{e}_x + az\sin\varphi'\vec{e}_y + a(a - r\cos(\varphi-\varphi'))\vec{e}_z}{(r^2+a^2+z^2-2ar\cos(\varphi-\varphi'))^{3/2}}$$

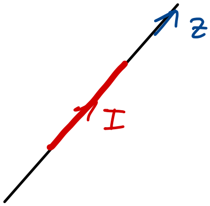
- na ose: $r = 0$:

$$\vec{B} = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{az\cos\varphi'\vec{e}_x + az\sin\varphi'\vec{e}_y + a^2\vec{e}_z}{(a^2+z^2)^{3/2}} d\varphi' =$$

$$= \frac{\mu_0}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\vec{e}_z$$

- pole solenoidu:

$$B = \frac{\mu_0 I a^2}{2}\int_{-L}^{L} \frac{dz'}{(a^2+(z-z')^2)^{3/2}} = \dots = \frac{\mu_0 I}{2}\left[\frac{z+L}{\sqrt{(z+L)^2+a^2}} - \frac{z-L}{\sqrt{(z-L)^2+a^2}}\right]$$

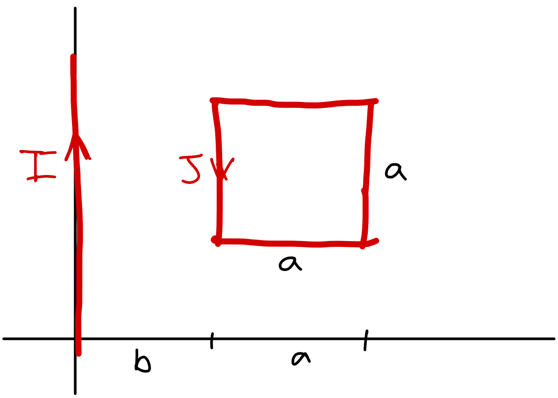


$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I\, d\vec{\ell}\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} =$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{-\vec{e}_x\, dz'\, r\sin\varphi + \vec{e}_y\, dz'\, r\cos\varphi}{\left(r^2+(z-z')^2\right)^{3/2}} =$$

$$\int_{R\,\mathrm{ctg}\,\alpha}^{R\,\mathrm{ctg}\,\beta} \frac{dz'}{\left(r^2+z'^2\right)^{3/2}} = \left[\frac{z'/r^2}{\sqrt{z'^2+r^2}}\right]_{R\,\mathrm{ctg}\,\alpha}^{R\,\mathrm{ctg}\,\beta} \Rightarrow \frac{1}{r}\left[\cos\beta-\cos\alpha\right]$$

$$\vec{r} = (r\cos\varphi,\ r\sin\varphi,\ z) \qquad d\vec{\ell} = (0,\ 0,\ dz')$$
$$\vec{r}\,' = (\ 0\ ,\ 0\ ,\ z')$$



$$\vec{B}_I = \frac{\mu I}{2}\,\frac{1}{R}\,\vec{e}_\varphi$$

$$\phi = \oint \vec{B}\cdot d\vec{S} =$$

$$= a\int_b^{b+a} \frac{\mu I}{2}\,\frac{1}{R}\, dR =$$

$$\phi_{vzaj} = \frac{a\mu_0 I}{2}\ln\frac{b+a}{b}$$

$$\Rightarrow L = \frac{a\mu_0}{2}\ln\frac{b+a}{b}$$

$$\phi_{vlast} = L J$$

$$U = RJ = -\frac{d\phi}{dt} = -L\frac{dJ}{dt} - M\dot{I} \Rightarrow L\dot{J} + RJ = -M\overset{\Theta(t-t_0)}{\dot{I}}$$

$$\Rightarrow J(t) = \cancel{C e^{-\frac{R}{L}t}}^{\,C=0} + \frac{I}{L}\,\Theta(t-t_0)\, e^{-\frac{R}{L}(t-t_0)}$$

ku přednášce

$$\vec m = \frac{1}{2}\int \vec x \times \vec j \, dV \qquad m^i = -\frac{1}{2}\,\varepsilon^{ijk} I_{jk}$$

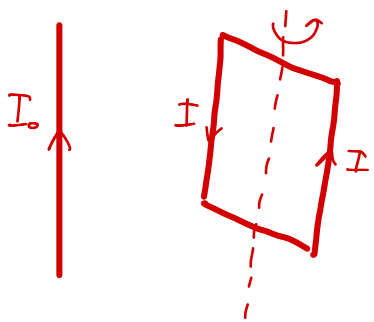
$$I_{jk} = \int x_j \, j_k \, dV$$

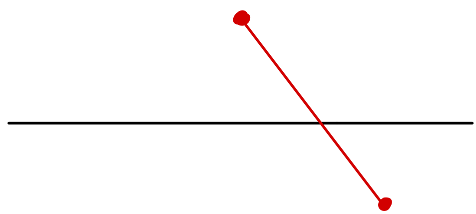
m je konst

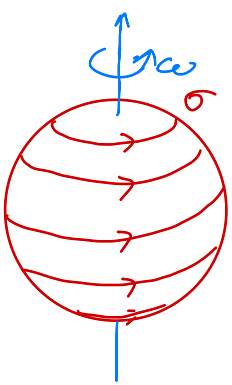
$$F_k = \frac{\partial B_i}{\partial x^k} m^i \iff F = \vec m \cdot \nabla \vec B = \nabla(\vec B \cdot \vec m)$$

$\nabla \times \vec B = 0 \iff$ symetrická část $\nabla \vec B = 0$

$$\vec M = \int \vec x \times (\vec j \times \vec B)\, dV = \underbrace{\int \vec j\, \vec x\, dV \cdot \vec B}_{I_{jk} B_k} - \overset{\text{antisymetrická } 0}{\cancel{\int \vec j \cdot \vec x\, dV}}\, \vec B = \vec m \times \vec B$$







$$[B_n] = 0$$

$$[\vec n \times \vec B] = \mu_0 \vec j$$

$$\vec j = \sigma \vec v = \sigma\, \vec\omega \times \vec r = \sigma\, \omega\, a \sin\theta\, \vec e_\varphi$$

$$\vec m = \frac{1}{2}\int \vec x \times \vec j \, dV = \frac{1}{2}\int \vec x \times \vec j \; a^2 \sin\theta \, d\Omega =$$

$$= \frac{1}{2} a^2 \int a\omega \sin^2\theta \; \vec e_r \times \vec e_\varphi \, d\Omega = \frac{a^3\omega}{2}\int \sin^2\theta \, \vec e_\theta \, d\Omega =$$

$$= \frac{a^3\omega}{2}\int \sin^2\theta\left(\sin\theta\,\vec{e}_z + \cos\theta\,\vec{e}_R\right) d\Omega =$$

$$= a^4\omega\pi \int_0^{\pi}\left(\sin^3\theta\,\vec{e}_z + \cancelto{0}{\cos\theta\sin^2\theta\,\vec{e}_R}\right) d\theta =$$

$$= a^4\omega\pi\int_0^{\pi}\sin^3\theta\, d\theta\, \vec{e}_z = a^4\omega\pi\,\frac{4}{3}$$

$$\Rightarrow \vec{m} = \frac{4}{3}\pi a^4\sigma\omega\,\vec{e}_z = \frac{Qa^2\omega}{3}\vec{e}_z$$

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left(3\vec{m}\cdot\vec{e}_r\vec{e}_r - \vec{m}\right) = \frac{\mu_0}{4\pi r^3}\left(Qa^2\omega\cos\theta\,\vec{e}_r - \frac{Qa^2\omega}{3}\vec{e}_z\right) =$$

$$= \frac{\mu_0 Q a^2\omega}{4\pi r^3}\left(\cos\theta\,\vec{e}_r - \frac{1}{3}\vec{e}_z\right)$$

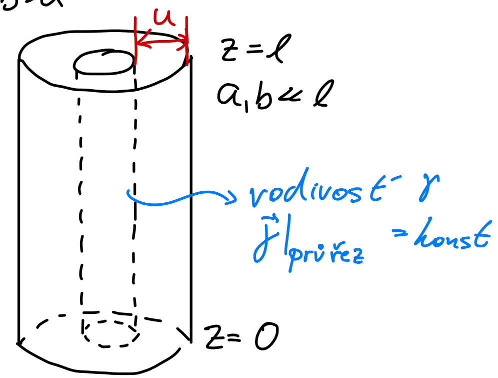
$$\vec{B}_n = \vec{e}_r\cdot\vec{B} = \frac{\mu_0 Q a^2\omega}{4\pi a^3}\,\frac{2}{3}\cos\theta = \frac{\mu_0 Q \;\omega\cos\theta}{6\pi a} = B_0\cos\theta$$

$$\Rightarrow B_0 = \frac{\mu_0 Q\omega}{6\pi a}$$

$$\vec{e}_r \times [\vec{B}] = \frac{\mu_0}{4\pi a^3}\left(\vec{e}_r\times\vec{m}\right) - B_0\,\vec{e}_z\times\vec{e}_r = \frac{\mu_0}{4\pi a^3}\,\frac{Qa^2\omega}{3} + \frac{\mu_0 Q\omega}{6\pi a}\left(\vec{e}_r\times\vec{e}_z\right)$$

$$=$$

$b > a$



$\varphi = ?$ $\vec{E} = ?$ $\vec{B} = ?$

$$\vec{E} = -\frac{U}{\ell}\vec{e}_z \qquad \vec{j} = -\frac{U}{\ell}\gamma\,\vec{e}_z = -\vec{e}_z\frac{I}{\pi a^2} \qquad I = \frac{\overbrace{\pi a^2 \gamma}^{\frac{1}{\text{odpor}}}}{\ell}U$$

$$\vec{j}_s = \vec{e}_z \frac{I}{2\pi a}$$

$$\vec{B}_{R>b} = 0 \qquad \vec{B}_{a<r<b} = -\frac{I}{2\pi}\mu_0 \frac{1}{R}\vec{e}_\varphi$$

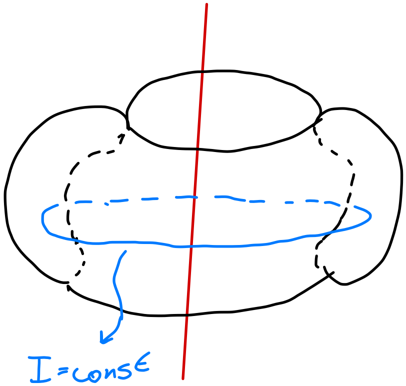
$$\vec{B}_{R<a} = \frac{-I\mu_0}{2\pi}\frac{R}{a^2}\vec{e}_\varphi$$

$$\vec{S}_{R<a} = \varepsilon_0 c^2\,\vec{E}\times\vec{B} = \frac{U}{\ell}\frac{I}{2\pi}\frac{R}{a^2}\vec{e}_R$$

$$\vec{S}_{R>b} = 0$$

$$\vec{S}_{a<R<b} = \varepsilon_0 c^2\,\vec{E}\times\vec{B} = \frac{U}{\ell}\left[-\frac{\vec{e}_R}{\log\frac{a}{b}}\frac{z}{R} - \frac{\log\frac{R}{b}}{\log\frac{a}{b}}\vec{e}_z\right]\times\left[-\frac{I}{2\pi}\frac{1}{R}\vec{e}_\varphi\right] =$$

$$= \frac{1}{2\pi}\frac{U}{\ell}\frac{-I}{\log\frac{b}{a}}\frac{z}{R^2}\vec{e}_z + \frac{\log\frac{R}{b}}{\log\frac{b}{a}}\frac{I}{2\pi R}\vec{e}_R$$



$$\frac{\partial}{\partial \varphi} = 0$$

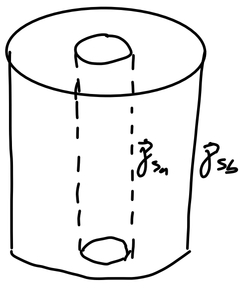
$$\vec{B} = \vec{B}(R, z)$$

$$\oint_{\partial\varphi} \vec{B}\cdot d\vec{l} = \mu_0 \int_{\varphi} \vec{j}\cdot d\vec{S} = I\mu_0$$

$$B_\varphi 2\pi R = I\mu_0 \Rightarrow B_\varphi = \frac{I}{2\pi R}\mu_0$$

$$\nabla\cdot\vec{B} = 0 \qquad \mathrm{Div}\,\vec{B} = 0 \quad \rightarrow \quad [\vec{B}\,\vec{n}] = \vec{B}_{in}\cdot\vec{n} = 0 \;\checkmark$$

$$\nabla\times\vec{B} = \underbrace{\vec{j}\mu_0}_{0} \qquad \mathrm{Rot}\,\vec{B} = \mu_0\vec{j}_s \quad \rightarrow \quad \vec{n}\times\vec{B}_{in} = \mu_0 j_s\vec{\tau} = \vec{n}$$



$$I_a = -I_b$$

$$\vec{B}_{r<a} = 0$$

$$\vec{B}_{r>b} = \frac{\mu_0}{r}(a j_{sa} + b j_{sb})\vec{e}_\varphi$$

$$\vec{B}_{a<r<b} = \mu_0 j_{sa}\frac{a}{r}\vec{e}_\varphi$$

περίπτωση $I_a = -I_b$

$$W_{mag} = \frac{1}{2\mu}\int B^2\, dV = \frac{1}{2\mu_0} 2\pi l \int_a^b \mu_0^2 j_{sa}^2 \frac{a^2}{r^2} r\,dr =$$

$$= \mu_0 \pi l a^2 j_{sa}^2 \ln\frac{b}{a} \qquad = \mu_0 \pi l j_{sa}^2 a^2 \ln\frac{b}{a}$$

$$\Rightarrow L = \frac{\mu_0 \pi l}{2\pi^2 a^2} a^2 \ln\frac{b}{a} = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$