Exam Kolman 03.06.2026.

1. a) State the precise definition of positive definite matrices **(5)**
   b) What do you know about the decomposition of positive definite matrices? State the result precisely and prove it. **(10)**

2. Decide whether the matrices $C = \begin{pmatrix} -7 & -4 & -8 \\ 4 & 3 & 4 \\ 8 & 4 & 9 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ -4 & -1 & -2 \\ 9 & 4 & 5 \end{pmatrix}$ are similar. **(15)**

3. a) Consider the vectors $b_1 = (1,0,0)^T, b_2 = (2,0,3)^T, b_3 = (4,5,6)^T$ in the vector space $V = \mathbb{R}^3$ with the standard inner product. Use Gram-Schmidt orthogonalization to obtain an orthonormal basis of V from basis (b_1, b_2, b_3) **(7)**
   b) Let $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{pmatrix}$. Compute a QR decomposition of A **(10)**

4. For each of the following statements, justify whether it is true or false.
   a) The determinant of the matrices $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 9 & 10 & 11 & 12 \\ 16 & 15 & 14 & 13 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ is the same. **(5)**
   b) For every two complex matrices A, B (of size $n \times n$): If $\lambda_1, ..., \lambda_n$ are the eigenvalues of matrix A and $\lambda'_1, ..., \lambda'_n$ are the eigenvalues of matrix B (taking each eigenvalue with its algebraic multiplicity), then $\lambda_1 \cdot \lambda'_1, ..., \lambda_n \cdot \lambda'_n$ are the eigenvalues of the matrix (AB). **(5)**
   c) The quadratic form $f(x) = x^T \begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix} x$ on $\mathbb{R}^2$ takes only negative values, with the exception of the vector $x = (0,0)^T$. **(5)**

Comments: Make sure to leave enough space as you are working, as no extra papers. The matrices in 2nd problem are really this (differing by one entry prior to last years)