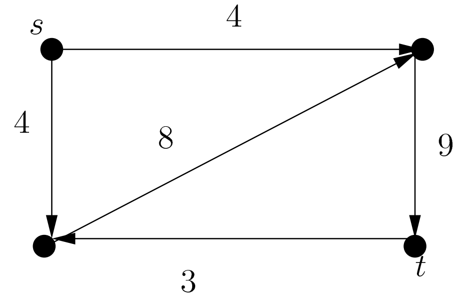

Figure 2.7. A cut of capacity 13

depending on whether F_1 keeps or interchanges the contact vertices. Analogously as above, applying at most $n - n_1 + 2 - 2$ twists to G'_{-1} we get graph G^* isomorphic to G. We used at most $(n_1 - 2) + (n - n_1) = n - 2$ twists.

□

In the next sections we study graph connectivity and network flows. The flows are flowing as much as we are walking in this book.

2.4 Flows in directed graphs

Let $D = (V, E)$ be a directed graph and let $v \in V$. A function f on E is a *flow at* v if it satisfies Kirchhoff's current law: the total flow into v is equal to the total flow leaving v, or $\sum_{(uv)\in E} f(uv) = \sum_{(vu)\in E} f(vu)$. A *circulation* is a flow at each vertex of D.
We study the flows from vertex s (the *source*) to vertex t (the *sink*). We say that a function f on E is an *s,t-flow*, if it is a flow at each vertex different from both s, t. Clearly, whichever flows out of s in an s,t-flow, must flow into t. Hence we may define the *value* $|f|$ of the s,t-flow f by

$$|f| = \sum_{(sv)\in E} f(sv) - \sum_{(us)\in E} f(us) = \sum_{(ut)\in E} f(ut) - \sum_{(tv)\in E} f(tv).$$

The Max-Flow Problem. We are given a directed graph $D = (V, E)$, two different vertices s, t of D and a non-negative rational *capacity* $c(e)$ of each edge $e \in E$. The max-flow problem is to find a maximum value s,t-flow f such that for each $e \in E$, $0 \le f(e) \le c(e)$.
If $S \subset V$ such that $s \in S$ and $t \notin S$ then let us denote by S^+ the set of directed edges leaving S, i.e., $S^+ = \{(u, v); u \in S, v \notin S\}$. Each such set S^+ is called an *s,t-cut*, and its capacity $c(S^+)$ is defined by $c(S^+) = \sum_{e\in S^+} c(e)$.

Theorem 2.4.1. *The maximum s,t-flow value is equal to the minimum capacity of an s,t-cut.*

Proof. We have $|f| = \sum_{(uv)\in S^+} f(uv) - \sum_{(vu)\in E, v\notin S, u\in S} f(vu)$ for each s,t-cut S^+ and so the maximum is at most the minimum.
In order to finish the proof, we update an s,t-flow f and subset $S \subset V$ recursively as follows. First let $S = \{s\}$ and f be the zero flow. If $u \in S$, $v \notin S$, and $c(u,v) > f(uv)$ or $f(vu) > 0$ then we add v to S. In the end two things may occur. If $t \in S$ then we can improve the value of f: let $s = v_0, v_1, \cdots, v_l = t$ be a path in S connecting s and t. We have

$$0 < \epsilon_i = \max\{(c(v_i, v_{i+1}) - f(v_i v_{i+1})), f(v_{i+1}, v_i)\}.$$

Let $\epsilon = \min \epsilon_i, i = 0, \cdots, l-1$. Then f may be *augmented* along that path to a flow f' with value $|f'| = |f| + \epsilon$ as follows: if $\epsilon_i = c(v_i v_{i+1}) - f(v_i v_{i+1})$ then let $f'(v_i v_{i+1}) = f(v_i v_{i+1}) + \epsilon$, else let $f'(v_{i+1} v_i) = f(v_{i+1} v_i) - \epsilon$.
On the other hand, if $t \notin S$, then f has maximum value since

$$|f| = \sum_{u\in S, v\notin S} f(uv) - \sum_{u\notin S, v\in S} f(uv) = \sum_{u\in S, v\notin S} c(uv) = c(S^+).$$

The capacities are rational and the graph is finite and so in each step we augment the flow by at least a fixed non-zero fraction. Hence a maximum flow is reached in a finite number of steps.

□

We remark that the proof also shows the following

Corollary 2.4.2. *If the capacity function is integral then there is a maximal flow that is integral.*

Let us assume now that we have capacity restrictions on the vertices, except for the source and the sink, and we want to find a maximum value flow f such that for each vertex v,

$$\sum_{(u,v)\in E} f(u,v) \le c(v).$$

This is reduced to the flows by an important operation called *splitting*.

Definition 2.4.3. *Let $D = (V, E)$ be a directed graph and let $v \in V$. We say that D' is obtained from D by splitting a vertex v if v is replaced by two new vertices v_s, v_t so that each edge of D starting in v starts in v_s of D', each edge of D terminating in v terminates in v_t of D', and $e(v) = (v_t, v_s) \in E(D')$.*

We say that a set U of vertices is an *s,t-vertex cut* if $D - U$ has no directed path from s to t.

Theorem 2.4.4. *Let D be a directed graph with a rational capacity $c(v) \ge 0$ on each vertex v different from s, t. Then the minimum capacity of an s,t-vertex cut is equal to the maximum s,t-flow value.*

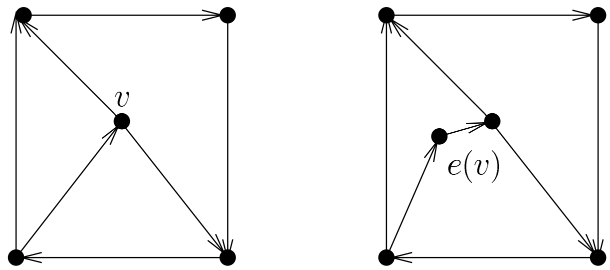

Figure 2.8. Splitting a vertex v

Proof. Let D' be obtained from D by splitting each vertex v different from s, t. We also let $c(v_t v_s) = c(v)$ and $c(e) = +\infty$ for all other edges of D'. Clearly, there is a bijection between the set of the flows of D and the set of the flows of D' and the s,t-cuts of D' of finite capacity exactly correspond to the s,t-vertex cuts. □

2.5 Connectivity

Let $G = (V, E)$ be a graph and let s, t be two distinct vertices of G. We say that two s,t-paths are *independent* if they have only the vertices s, t in common. The following Menger's theorem is perhaps the most important theorem on the graph connectivity.

Theorem 2.5.1. *Let s and t be distinct non-adjacent vertices of a graph G. Then the minimum number of vertices separating s from t is equal to the maximum number of independent s,t-paths. Moreover, if s, t are distinct vertices of G then the minimum number of edges separating s from t is equal to the maximum number of edge-disjoint s,t-paths.*

Proof. This follows from the max-flow min-cut theorems 2.4.1, 2.4.2 and 2.4.4 if we replace each edge by two oppositely oriented directed edges with the same endvertices, and let the capacities of the vertices (for the first statement) and of the edges (for the second statement) be 1. □

Corollary 2.5.2. *For $k \geq 2$, a graph is k-connected if and only if it has at least 2 vertices and any two vertices may be joined by k independent paths. Also, a graph is k-edge-connected if and only if it has at least two vertices and any two vertices may be joined by k edge-disjoint paths.*

The last observation on connectivity characterises 2-edge-connectivity by the existence of an orientation. The relations of connectivity and the existence of orientations with particular properties is a flourishing research subject with many results much stronger than the one we present below.

We say that a directed graph is *strongly connected* if there is a directed path from each vertex to any other vertex.

1920 to explain ferromagnetism and was named after his student Ising who contributed to its theory. The idea is that iron atoms are situated at vertices of a graph $G = (V, E)$ and behave like small magnets that can be oriented upwards (have *spin* $+1$) or downwards (have *spin* -1). Physically, two magnets that are close to one another need less energy to be oriented in the same way than in the opposite way. This leads to the following simplified model:
We assume that each edge uv of G has an assigned weight (or, a *coupling constant*) $w(uv)$. A configuration of the system is an assignment of the *spin* $\sigma_v \in \{+1, -1\}$ to each vertex v. This describes the two possible spin orientations the vertex can take. As is custumary in statistical physics, we will call the configurations of the Ising model its *states*.
The *energy function* (or *Hamiltonian*) of the model is defined as

$$z(\sigma) = -\sum_{uv \in E} w(uv)\sigma_u \sigma_v.$$

We can rewrite the Hamiltonian in the following way:

$$z(\sigma) = \sum_{\{u,v\} \in C} w(uv) - \sum_{\{u,v\} \in E \setminus C} w(uv) = 2w(C) - W,$$

where C is the set of edges connecting the pairs of vertices of different spins, and $W = \sum_{\{u,v\} \in E} w(uv)$ is the sum of all edge weights in the graph. Clearly, if we find the value of max-cut (see Section 2.3 for the definition of max-cut, min-cut), we have found the maximum energy of the model. Similarly, min-cut corresponds to the minimum energy (groundstate). The partition function is defined by

$$Z(G, \beta) = \sum_{\sigma} e^{-\beta z(\sigma)}.$$

In the next section we describe an efficient algorithm to determine the groundstate energy of the Ising model for any planar graph. In fact the whole partition function may be determined efficiently for planar graphs, and the principal ingredient is the concept of enumeration dualities of Section 6.4 and Section 6.3.

6.2 Max-Cut for planar graphs

Observation 2.10.5 reduces, for planar graphs, the Max-Cut problem to the *Maximum even subset problem.* Given a graph $G = (V, E)$ with rational weights on the edges, find the maximum value of $\sum_{e \in E'} w(e)$ over all even subsets $E' \subset E$. The following theorem thus means that the Max-Cut problem is efficiently solvable for planar graphs.

Theorem 6.2.1. *The maximum even subset problem is efficiently solvable for general graphs.*

The theorem is proved by a reduction to the *matching problem*. An efficient algorithm to find a *maximum matching* of a graph has been found by Edmonds (see Section 2.6). Edmonds also found an efficient algorithm to solve the
Weighted perfect matching problem: Given a graph $G = (V, E)$ with rational weights on the edges, the weighted perfect matching problem asks for the maximum value of $\sum_{e \in E'} w(e)$ over all perfect matchings E' of G.
This algorithm together with the following reduction of Fisher proves Theorem 6.2.1.

Theorem 6.2.2. *Given a graph $G = (V, E)$ with a weight function w on E, it is possible to construct graph $G' = (V', E')$ and a weight function w' on E' so that there is a natural weight preserving bijection between the set of the even sets of G and the set of the perfect matchings of G'.*

Proof. The graph G' may be constructed from G by a local transformation at each vertex, described in Figure 6.1. We let $w'(e) = w(e)$ for all edges e of G, and $w'(e) = 0$ for each new edge. It is straightforward to verify the statement

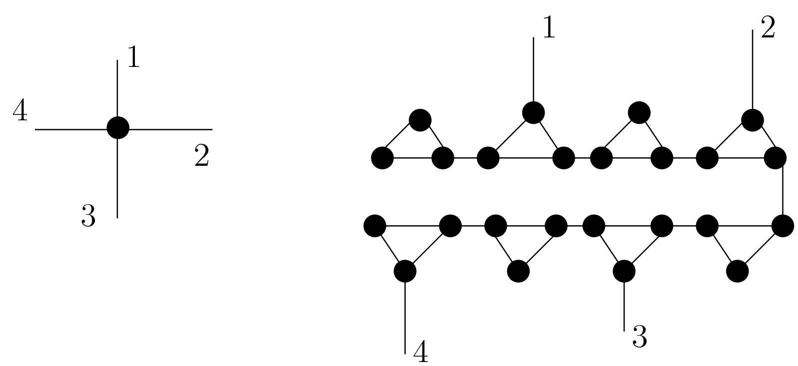

Figure 6.1. Each vertex is replaced by a path of triangles

of the theorem for G and G'. □

In statistical physics, a perfect matching is known under the name of a *dimer arrangement* or a dimer configuration. The *dimer problem* is the problem to enumerate all perfect matchings, i.e., to find the *partition function*:

$$\mathcal{P}(G, x) = \sum_{E' \text{ perfect matching}} x^{\sum_{e \in E'} w(e)}.$$

In discrete mathematics, partition functions are called *generating functions*. The bijection of Theorem 6.2.2 does more than just a reduction of the maximum even set to the maximum perfect matching. The whole information about the even sets is transformed to the information about the perfect matchings. This may be expressed by the equality of the generating functions.

Let G be a graph and let w be a weight function on E. For $E' \subset E$ let $w(E') = \sum_{e \in E'} w(e)$. We define the generating functions $\mathcal{C}(G,x), \mathcal{E}(G,x), \mathcal{P}(G,x)$ of the edge-cuts, the even sets and the perfect matchings as follows.

$$\mathcal{C}(G,x) = \sum_{E' \text{ edge-cut}} x^{w(E')},$$

$$\mathcal{E}(G,x) = \sum_{E' \text{ even set}} x^{w(E')},$$

$$\mathcal{P}(G,x) = \sum_{E' \text{ perfect matching}} x^{w(E')}.$$

The transformation of Theorem 6.2.2 shows that $\mathcal{E}(G,x) = \mathcal{P}(G',x)$.

6.3 Van der Waerden's theorem

The Ising partition function for a graph G may be expressed in terms of the generating function of the even sets of the same graph G. This is a theorem of Van der Waerden. We will use the following standard notation: $\sinh(x) = (e^x - e^{-x})/2$, $\cosh(x) = (e^x + e^{-x})/2$, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$.

Theorem 6.3.1.

$$Z(G,\beta) = 2^{|V|} \left(\prod_{uv \in E} \cosh(\beta w(uv)) \right) \left(\sum_{E' \subset E \ \mathit{even}} \prod_{uv \in E'} \tanh(\beta w(uv)) \right).$$

Proof. We have

$$Z(G,\beta) = \sum_\sigma e^{\beta \sum_{uv} w(uv) \sigma_u \sigma_v} =$$

$$\sum_\sigma \prod_{uv \in E} (\cosh(\beta w(uv)) + \sigma_u \sigma_v \sinh(\beta w(uv))) =$$

$$\prod_{uv \in E} \cosh(\beta w(uv)) \sum_\sigma \prod_{uv \in E} (1 + \sigma_u \sigma_v \tanh(\beta w(uv))) =$$

$$\prod_{uv \in E} \cosh(\beta w(uv)) \sum_\sigma \sum_{E' \subset E} \prod_{uv \in E'} \sigma_u \sigma_v \tanh(\beta w(uv)) =$$

$$\prod_{uv \in E} \cosh(\beta w(uv)) \sum_{E' \subset E} (U(E') \prod_{uv \in E'} \tanh(\beta w(uv))),$$

where

$$U(E') = \sum_\sigma \prod_{uv \in E'} \sigma_u \sigma_v.$$

The proof is complete when we notice that $U(E') = 2^{|V|}$ if E' is even and $U(E') = 0$ otherwise. $\square$