Algebra I

22.01.2026

2.5 hours

Grading system: 0-2 pt = "4", 3-5 pt = "3", 6-8 pt = "2", 9-11 pt = "1"

Exercise 1 (1pt) Let $\mathbf{A}$ be a ring with 1 and let $a, b \in A$. Show that

$$(1-ab)c = 1 = c(1-ab) \implies (1-ba)d = 1 = d(1-ba),$$

where $d = 1 + bca$, that is, $1 - ba$ is invertible in $\mathbf{A}$ whenever $1 - ab$ is invertible. Calculate the value of $1 + adb$.

Exercise 2 (1pt) Find a graph G with an automorphism group $\operatorname{Aut} G \cong \mathbb{Z}_3$.

Exercise 3 (1pt) Find the greatest common divisor of $f = 6x^3 - 18x^2 - 156x - 84$ and $g = 4x^3 - 16x^2 - 88x + 28$ in $\mathbb{Z}[x]$.

Exercise 4 (1pt+1pt) For two subgroups $\mathbf{H}, \mathbf{K}$ of a group $\mathbf{G}$, prove that $\mathbf{H} \cdot \mathbf{K} \leq \mathbf{G}$ if and only if $\mathbf{H} \cdot \mathbf{K} = \mathbf{K} \cdot \mathbf{H}$. Give an example of a group $\mathbf{G}$ and its subgroups $\mathbf{H}, \mathbf{K}$ such that $\mathbf{H} \cdot \mathbf{K}$ is not a subgroup of $\mathbf{G}$.

Exercise 5 (2pt) Show that every ring, where each element x satisfies $x^2 = x$, is commutative.

Exercise 6 (1pt+1pt) Show that a ring $\mathbf{A} = \mathbb{Z}_3[x]/(x^3 + 2x + 1)$ is a field and compute $(x+2)^{80}$ in $\mathbf{A}$.

Exercise 7 (2pt) Determine all subgroups of $\mathbf{D}_8$ and their sizes. Draw a poset of subgroups ordered by inclusion.