ALGEBRA I

11.02.2026

2.5 hours

Grading system: 0-2 pt = "4", 3-5 pt = "3", 6-8 pt = "2", 9-11 pt = "1"

EXERCISE 1 (1PT) Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

EXERCISE 2 (1PT) Find the greatest common divisor of $a = 11+3i$ and $b = 1+8i$ and the corresponding Bézout coefficients in $\mathbb{Z}[i]$ with Euclidean norm $\nu(a+ib) = |a^2+b^2|$.

EXERCISE 3 (1PT) Let S_8 be the group of all permutations on 8 elements and $\sigma = (1\,2) \in S_8$. Describe the set of all elements $\tau \in S_8$, such that $\tau\sigma = \sigma\tau$. Is it a subgroup of S_8?

EXERCISE 4 (1PT) Find all solutions of the following system of equations over $\mathbb{Z}$:

$$\begin{cases} x^2+2x+6 \equiv 0 \pmod 7, \\ 3x \equiv 2 \pmod 8. \end{cases}$$

EXERCISE 5 (1PT+1PT) Let $f \in \mathbb{Z}_p[x]$ be a polynomial where p is prime. We define a formal derivative f' in the usual way, e.g. if $f = x^n$, then $f' = nx^{n-1}$. Prove that:

- if $(x-a)^2$ divides f, then $f'(a) = 0$,
- $x^p - x$ has no multiple roots in any extension of $\mathbb{Z}_p$.

EXERCISE 6 (2PT) Let $\mathbb{Z}_{(2)} = \left\{\frac{a}{b} \in \mathbb{Q} \colon a, b \in \mathbb{Z} \text{ and } \mathrm{GCD}(a,b) = 1 \text{ and } 2 \nmid b\right\}$ be a ring of all rational numbers, where the denominator is odd with the standard addition and multiplication. Prove that it is a Euclidean domain with the norm $\nu\left(2^k \frac{a}{b}\right) = k \in \mathbb{Z}$, where $\mathrm{GCD}(a,b) = 1$ and $2 \nmid a, b$.

identity element

EXERCISE 7 (1PT) Let G be a finite abelian group, p be a prime number and $I_p = \{a \in G \colon a^p = 1\}$. Show that I_p forms a subgroup of G.

EXERCISE 8 (2PT) Let M be a 3×2 matrix with entries from the set $\{0, 1, 2\}$. Two matrices are considered equivalent if one can be obtained from the other through a sequence of row and column permutations. How many inequivalent matrices exist?