Introduction to Category Theory

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Introductory Note

The lecturer, non other than Dr Sharoch, may be contacted at the address `saroch@karlin.mff.cuni.cz`.

Examination shall be oral. Since Dr Sharoch will be absent for most of the examination-period, most examination-dates will be at the beginning and end of the period. One date in September is gauranteed.

Recommended Literature

Some of these titles can be found at the `library genesis` website.

- Eilenberg, Steenrod: *Foundations of Algebraic Topology* (1952), Princeton University Press.
- B. Mitchell: *Theory of Categories* (1965), Academic Press.
- H. Herrlich, G. E. Strecker: *Category Theory* (1973), Allyn & bacon.
- J. Adamek, H. Herrlich, Strecker: *Abstract & Concrete Categories* (1990), J. Wiley & Sons.
- M. Barr, C. Wells: *Toposes, Triples, Theories* (1985), Springer.
- S. MacLane: *Categories for Working Mathematicians* (1971), Springer-Verlag.

Dr Sharoch recommends the second and fourth items on this list.

Contents

1 Categories & Morphisms

The notion of 'category' first appeared in 1945 in an article of S. MacLane and S. Eilenberg: *On Natural Equivalences* in the publication *Transactions of AMS*.

Definition 1.1 (Category)**.** Let $\operatorname{Obj}\mathcal{K}$ be a class and $\mathcal{K}(a,b)$ be a set for any $a, b \in \operatorname{Obj}\mathcal{K}$. The former is termed a class of *objects* and the latter a set of *morphisms* or *arrows* from a to b. For any $\alpha \in \mathcal{K}(a,b)$, a is termed the *domain* of α and b the *codomain* of α, denoted $\operatorname{Dom}\alpha$ and $\operatorname{Cod}\alpha$ respectively.

Put

$$\operatorname{Mor}\mathcal{K} = \bigcup_{(a,b)\in(\operatorname{Obj}\mathcal{K})^2} \mathcal{K}(a,b).$$

A *category* is any interpretation of $\mathcal{K} = (\operatorname{Obj}\mathcal{K}, \operatorname{Mor}\mathcal{K})$ which in addition meets the following axioms:

(1) *Morphism Disjointness.*

$$(\forall a, b, a', b' \in \operatorname{Obj}\mathcal{K})\,(a,b) \neq (a',b') \Rightarrow \mathcal{K}(a,b) \cap \mathcal{K}(a',b') = \varnothing.$$

(2) *Composition.*

$$(\forall \alpha \in \mathcal{K}(a,b), \beta \in \mathcal{K}(b,c)\,\exists!\beta)\,\beta \circ \alpha \in \mathcal{K}(a,c).$$

(3) *Associativity.*

$$(\forall \alpha \in \mathcal{K}(a,b), \beta \in \mathcal{K}(b,c), \gamma \in \mathcal{K}(c,d))\,\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha.$$

(4) *Unit Law.*

$$(\forall \alpha \in \operatorname{Obj}\mathcal{K}\,\exists 1_a \in \mathcal{K}(a,a) \forall \beta \in \mathcal{K}(a,b) \forall \gamma \in \mathcal{K}(c,a))\,(\beta \circ 1_a = \beta) \wedge (1_a \circ \gamma = \gamma).$$

Remark 1.2. The morphism 1_a from **??** is uniquely determined by its properties. If $1'_a$ were another such morphism, then $1'_a \circ 1_a = 1'_a$.

Definition 1.3 (Locally Small Category)**.** Sometimes the defnition admits $\mathcal{K}(a,b)$ being a class, and not just a set. The kinds of categories satisfying our restricted definition is then called *locally small*.

Notation 1.4. If $\mathcal{K}$ is clear from context, instead of $\alpha \in \mathcal{K}(a,b)$ we sometimes write $\alpha : a \to b$. Another standard notation is $\langle \alpha, a, b \rangle$.

Example 1.5.

(1) $\mathcal{SET}$ denotes the category of all sets. Here objects are sets and morphisms are set mappings with a specified domain and codomain.

Let $A, B^* \subsetneq B$ be sets, $f : A \to B$, $f^* : A \to B^*$ and $(\forall a \in A)\,f^*(a) = f(a)$.

Then $f \neq f^*$ in the category $\mathcal{SET}(\mathcal{SET}(A,B) \cap \mathcal{SET}(A,B^*) = \varnothing$ even though $f = f^*$ componentwise.

(2) *Structured Sets.*

- $\mathcal{POSET}$ is a category whose objects are posets, i.e. partially ordered sets; and whose morphisms are monotonic maps.

  This means that $\alpha : (a, \leqslant) \to (b, \leqslant)$ implies that for each $x, y \in a$, $x \leqslant y \Rightarrow \alpha(x) \leqslant \alpha(y)$.

- $\mathcal{QOSET}$ is a category whose objects are quasiordered sets and whose morphisms are as above.

- $\mathcal{GRAPH}$ is a category whose objects are oriented graphs (i.e. the set V of all vertices and a relation A of two vertices being connected by an oriented edge) and whose morphisms are graph-homomorphisms.

  To clarify: let $G = (V, A)$, $G'(V', A')$, $\alpha : G \to G'$, $\alpha : (\forall v, w \in V)$. Then

  $$(v, w) \in A) \Rightarrow (\alpha(v), \alpha(w)) \in A'.$$

- $\mathcal{GRP}$ is a category whose objects are groups and morphisms are group-homomorphisms.

  Let $(G, \cdot), (H, *)$ be groups. Then $\alpha : (G, \cdot) \to (H, *)$ implies $\alpha(g_1 \cdot g_2) = \alpha(g_1) * \alpha(g_2)$.

- $\mathcal{SMG}$ is a category of semigroups (sets equipped with an associative operation): its objects are semigroups and its morphisms are semigroup homomorphisms

- $\mathcal{RNG}$ is a category whose objects are unitary rings and whose morphisms are ring-homomorphisms.

- $\mathcal{TOP}$ is a category whose objects are topological spaces and whose morhpisms are continuous maps.

(3) *Arbitrary Examples.*

- Let $\operatorname{Obj} \mathcal{K}$ be the elements of $\mathbb{R}^2$ and $\operatorname{Mor} \mathcal{K}$ be finite polygonal lines connecting its two arguments.

Remark 1.6. Given a general category $\mathcal{K}$ it is not meaningful to consider the elements of the objects of a category $\mathcal{K}$.

Definition 1.7. A category $\mathcal{K}$ is said to be *thin* if for every $a, b \in \operatorname{Obj} \mathcal{K}$,

$$|\,\mathcal{K}(a, b)| \leqslant 1.$$

Remark 1.8. Thin categories correspond one-to-one with quasiordered classes: Suppose $\mathcal{K}$ is a category. Define the ordering $\leqslant$ on $\operatorname{Obj} \mathcal{K}$ by

$$a \leqslant b \Leftrightarrow |\,\mathcal{K}(a, b)| = 1.$$

Then

$$\mathcal{K} \hookrightarrow (\operatorname{Obj}\mathcal{K}, \leqslant).$$

Conversely, given a quasiordering Q, let $\operatorname{Obj}\mathcal{K} = Q$ and for any two $a, b \in Q$, let there exist a single morphism between them if $a \leqslant b$. If $\neg(a \leqslant b)$, then let there be no such morphism.

Clearly

$$\mathcal{K} \hookleftarrow (Q, \leqslant).$$

Definition 1.9 (Discrete Category)**.** A category $\mathcal{K}$ is said to be *discrete* if it contains only identity-morphisms. Symbolically

$$|\,\mathcal{K}(a,b)| = \delta_{a,b}.$$

Definition 1.10 (Small Category)**.** A category $\mathcal{K}$ is said to be *small* if $\operatorname{Obj}\mathcal{K}$ is a set (as opposed to a class).

Remark 1.11. Singleton categories correspond one-to-one with monoids. For any category $\mathcal{K}$ with a single object A, the set $\mathcal{K}(A, A)$ of morphisms $A \to A$ is a monoid. with unit element Id_A and a binary operation given by composition; conversely for any monoid $(M, \cdot, e)$, there is a one-object category $\mathcal{K}$ with a single object $\operatorname{Obj}\mathcal{K} = \{M\}$ whose morphisms are the elements of M, with $\mathrm{Id}_A = e$ and such that composition is given by '$\cdot$'.

$$\begin{aligned} \mathcal{K} &\hookrightarrow (\mathcal{K}(A,A), \circ, \mathrm{Id}_A) \\ (\{M\}, M) &\hookleftarrow (M, \cdot, e) \end{aligned}$$

Definition 1.12 (Monomorphism, Epimorphism, Bimorphism, Isomorphism)**.** A morphism $\alpha \in \mathcal{K}(a,b)$ is called a *monomorphism* if, given any two morphisms $\gamma, \beta : c \to a$; $\alpha \circ \beta = \alpha \circ \gamma$ implies $\beta = \gamma$.

Dually, $\alpha \in \mathcal{K}(a,b)$ is called an *epimorphism* if, given $\beta, \gamma \in \mathcal{K}(b,c)$, $\beta \circ \alpha = \gamma \circ \alpha$ implies $\beta = \gamma$.

If α is both monic and epic, then it is termed a *bimorphism*.

Lastly, $\alpha \in \mathcal{K}(a,b)$ is called an *isomorphism* if there exists $\beta \in \mathcal{K}(b,a)$ such that $\beta \circ \alpha = 1_a$, $\alpha \circ \beta = 1_b$

Definition 1.13 (Balanced Category)**.** A category is called *balanced* if a morphism is epic and monic iff it is an isomorphism.

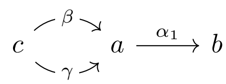
Theorem 1.14 (Heredity of Moncity & Epicity)**.**

(1) If $\alpha_1 \in \mathcal{K}(a,b)$, $\alpha_2 \in \mathcal{K}(b,c)$ are monic, then $\alpha_1 \circ \alpha_2$ is likewise monic.

(2) Conversely, if $\alpha_2 \circ \alpha_1$ are monic, then so is α_1.

(3) Dually, the analogous propositions hold for epimorphisms.

Proof. Trivial. Only point (2) shall be shown. Assume



and let $\alpha_1 \circ \beta = \alpha_1 \circ \gamma$. Then $(\alpha_2 \circ \alpha_1) \circ \beta = (\alpha_2 \circ \alpha_1) \circ \gamma$. Since $\alpha_2 \circ \alpha_1$ is monic, $\beta = \gamma$. QED

Definition 1.15 (Opposite Category)**.** The category $\mathcal{K}^{\mathrm{op}}$ denotes a category *dual* to $\mathcal{K}$, also called *opposite*. We have $\mathrm{Obj}\,\mathcal{K}^{\mathrm{op}} := \mathrm{Obj}\,\mathcal{K}$, $\mathcal{K}^{\mathrm{op}}(a,b) = \mathcal{K}(b,a)$, for all $a, b \in \mathrm{Obj}\,\mathcal{K}$.

Remark 1.16. Monomorphisms in $\mathcal{K}^{\mathrm{op}}$ correspond 1-to-1 with epimorphisms in $\mathcal{K}$.

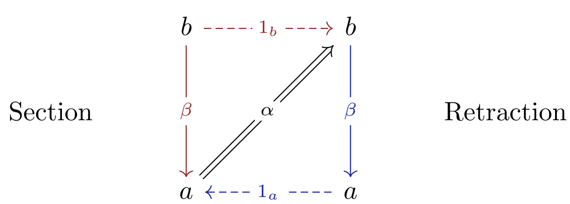
Theorem 1.17 (Duality Priniciple)**.** *Isomorphisms are monic and epic.*[1]

Proof. Let $\alpha \in \mathcal{K}(a,b)$ be an isomorphism and let $\beta \in \mathcal{K}(b,a)$ be a morphism witnessing the isomorphic property of α.

If $\gamma, \delta : c \to a$ are any morphisms such that $\alpha \circ \delta = \alpha \circ \gamma$, then $(\beta \circ \alpha) \circ \delta = (\beta \circ \alpha) \circ \gamma$ implying $1_a \circ \delta = 1_a \circ \gamma$ whence $\gamma = \delta$. That is to say, α is epic. To see it is also monic, let $\gamma, \delta : b \to c$ and argue by analogy. QED

Definition 1.18 (Section, Retraction)**.** A morphism $\alpha \in \mathcal{K}(a,b)$ for which there exists $\beta \in \mathcal{K}(b,a)$ such that $\beta \circ \alpha = 1_a$ is called a *split monomorphism* or *section*; if $\alpha \circ \beta = 1_b$, it is termed a *split epimorphism* or *retraction*.

More bluntly, a section is a right inverse and a retraction a left inverse of a morphism.



Definition 1.19 (Witness)**.** Let $\mathcal{A}$ be a first-order language, $\phi(\overline{v})$ be an $\mathcal{A}$-formula, and $\boldsymbol{A}$ be an interpretation of $\mathcal{L}$. An $\boldsymbol{A}$-*witness*, or simply a *witness* if $\boldsymbol{A}$ is clear from context, for the sentence $\exists \overline{v}\, \phi(\overline{v})$ is an element $\overline{a} \in \boldsymbol{A}^{\mathrm{len}\,\phi}$ such that $\boldsymbol{A} \models \phi(\overline{a})$.

For our purposes, by a *witness* we shall understand any object, morphism, category, etc. whose existence proves a particular property. Given a map $\alpha \in \mathcal{K}(a,b)$, the morphism $\beta \in \mathcal{K}(b,a)$ is a witness of α being a section if $\beta \circ \alpha = 1_b$.

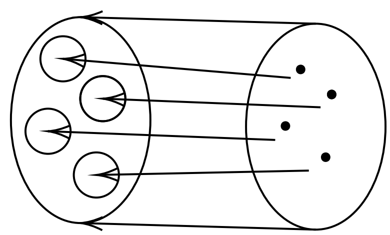
[1]Observe the converse does not generally hold; in view of this theorem, one could define balanced categories as those categories which satisfy such (converse) implication.

Remark 1.20. Clearly any section is monic and conversely any retraction is epic. A morphism which is both a section and a contraction is an isomorphism.

Example 1.21.

(1) In $\mathcal{SET}$, morphisms are monic iff they are injective and epic iff they are surjective. The category $\mathcal{SET}$ is balanced.

One may prove that the Axiom of Choice holds iff every epimorphism in $\mathcal{SET}$ is split (a retraction).



(2) In the balanced categories, $\mathcal{GRP}$, $\mathcal{AB}$, and the category of right $\boldsymbol{R}$-modules, monomorphisms are precisely injective homomorphisms and epimorphisms are surjective homomorphisms.

Showing this for $\mathcal{GRP}$ is nontrivial.

(3) In the category of polygonal lines, a morphism if monic iff it is epic iff it is the identity-morphism 1_B for $B \in \mathbb{R}^2$.

(4) The category of divisible Abelian groups $\mathcal{DAB}$ and the category of torsionfree Abelian groups $\mathcal{TFAB}$ whose morphisms are homomorphisms of Abelian groups.

The category $\mathcal{DAB}$ is not balanced and neither is $\mathcal{TFAB}$ since the monic canonical embedding $\nu : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism.

$$\mathbb{Z} \xhookrightarrow{\ \nu\ } \mathbb{Q} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B \in \mathcal{TFAB}$$

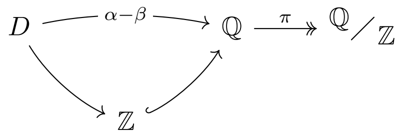
$\alpha \circ \nu = \beta \circ \nu$ implies $\alpha - \beta \restriction \mathbb{Z} = 0$, which means there exists $\gamma : \mathbb{Q}/\mathbb{Z} \to B$; then $\gamma = 0$ since $\operatorname{Cod}\gamma$ is torsionfree and $\operatorname{Dom}\gamma$ is not, and hence $\gamma \circ \pi = \alpha - \beta = 0$ for $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$.

(5) In $\mathcal{RNG}$, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is also an epimorphism (exercise).

Remark 1.22. The homomorphic image of a divisible group is divisible and any subgroup of a torsionfree group is torsionfree.

Moreover, in $\mathcal{DAB}$ the map $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is a monomorphism.

Suppose $\pi \circ \alpha = \pi \circ \beta$, then $\pi \circ (\alpha - \beta) = 0$. This implies that $\operatorname{Im}(\alpha - \beta) \subset \operatorname{Ker}(\pi) = \mathbb{Z}$. Since $\operatorname{Im}(\alpha - \beta)$ is divisible, then $\operatorname{Im}(\alpha - \beta)$ being a subset of $\mathbb{Z}$ implies $\operatorname{Im}(\alpha - \beta) = \{0\}$.



The fact $\mathcal{DAB}$ is not balanced follows hence given that $\pi : \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is not an isomorphism.

Notation 1.23. Recall $\hookrightarrow$ denotes a monomorphism and $\twoheadrightarrow$ an epimorphism.

2 Functors

Definition 2.1 (Covariant Functor)**.** Let $\mathcal{K}$, $\mathcal{H}$ be categories. It is said $F : \mathcal{K} \to \mathcal{H}$ is a *covariant functor* from $\mathcal{K}$ to $\mathcal{H}$ if for each $a \in \operatorname{Obj}\mathcal{K}$ and $\alpha \in \mathcal{K}(a, b)$, $\beta \in \mathcal{K}(b, c)$:

(1) $F(a) \in \operatorname{Obj}\mathcal{H}$

(2) $F(\alpha) \in \mathcal{H}(F(a), F(b))$

(3) $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$

(4) $F(1_a) = 1_{F(a)}$

Definition 2.2 (Contravariant Functor)**.** Let $\mathcal{K}$, $\mathcal{H}$ be categories. It is said $F : \mathcal{K} \to \mathcal{H}$ is a *contravariant functor* from $\mathcal{K}$ to $\mathcal{H}$ if for each $a \in \operatorname{Obj}\mathcal{K}$ and $\alpha \in \mathcal{K}(a, b)$, $\beta \in \mathcal{K}(b, c)$:

(1) $F(a) \in \operatorname{Obj}\mathcal{H}$

(2) $F(\alpha) \in \mathcal{H}(F(b), F(a))$

(3) $F(\beta \circ \alpha) = F(\alpha) \circ F(\beta)$

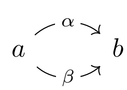
(4) $F(1_a) = 1_{F(a)}$

Note that contravariant functors reverse the direction of composition.

Remark 2.3. In the topological category $\mathcal{TOP}$ monomorphisms are injective morphisms and epimorphisms are surjective morphisms. A morphism which is monic and epic is a continuous bijection, which is not the same as a homeomorphism, which is an isomorphism in the categorical sense.

To illustrate this, consider a set X with at least two elements, consider the map $(X, \text{discrete}) \xrightarrow{\text{Id}} (X, \text{indiscrete})$.

Definition 2.4 (Parallel Morphism)**.** Let $\alpha, \beta \in \mathcal{K}(a, b)$, then α, β are termed *parallel morphisms.*



Definition 2.5 (Homfunctor)**.** Let $\mathcal{K}$ be a (locally small) category, $a, b \in \operatorname{Obj}\mathcal{K}$ and $\alpha \in \mathcal{K}(b, c)$. We introduce the functions

$$\begin{aligned} \mathcal{K}(a,\alpha) : \mathcal{K}(a,b) &\to \mathcal{K}(a,c) \qquad & \mathcal{K}(\alpha,a) : \mathcal{K}(c,a) &\to \mathcal{K}(b,a) \\ \beta &\mapsto \alpha \circ \beta & \beta &\mapsto \beta \circ \alpha. \end{aligned}$$

Then $\mathcal{K}(a, -)$ is said to be a covariant homfunctor of $\mathcal{K}$ if

(1) $\mathcal{K}(a, -) : b \mapsto \mathcal{K}(a, b)$.

(2) $\mathcal{K}(a, -) : \alpha \mapsto \mathcal{K}(a, \alpha)$.

Likewise, $\mathcal{K}(-, a)$ is said to be a contravariant homfunctor if

(1) $\mathcal{K}(-, b) : a \mapsto \mathcal{K}(a, b)$.

(2) $\mathcal{K}(-, b) : \alpha \mapsto \mathcal{K}(\alpha, a)$.

Remark 2.6. The covariant and contravariant functors are naturally related. Let $a, b, a', b' \in \operatorname{Obj}\mathcal{K}$ and $f \in \mathcal{K}(a, a')$, $h \in \mathcal{K}(b, b')$. Then

$$\begin{array}{ccc} \mathcal{K}(a,b) & \xrightarrow{\mathcal{K}(h,b)} & \mathcal{K}(a',b) \\ \Big\downarrow{\scriptstyle \mathcal{K}(a,f)} & & \Big\downarrow{\scriptstyle \mathcal{K}(a',f)} \\ \mathcal{K}(a,b') & \xrightarrow{\mathcal{K}(h,b')} & \mathcal{K}(a',b') \end{array}$$

We shall learn more about this later when discussing natural transformations.

Example 2.7.

(1) *Forgetful Functors.* Functors are termed *forgetful* if they, in a sense, 'forget' part of the structure they are defined on, e.g.

$$\begin{aligned} F &: \mathcal{TOP} \to \mathcal{SET} \qquad & (X, \tau) &\to X \\ F &: \mathcal{GRP} \to \mathcal{SET} & (G, \cdot) &\mapsto G. \end{aligned}$$

Another example

$$\begin{aligned} F &: \mathcal{TOPGRP} \to \mathcal{TOP} \\ F &: \mathcal{TOPGRP} \to \mathcal{GRP}. \end{aligned}$$

(2) Let $\mathcal{K}$ be a category and $a \in \operatorname{Obj}\mathcal{K}$. Let $F_a : \mathcal{K} \to \mathcal{SET}$; for some $b \in \operatorname{Obj}\mathcal{K}$, $F_a(b) := \mathcal{K}(a, b) \in \operatorname{Obj}\mathcal{SET}$. Likewise for $\alpha : c \to b$, $F_a(\alpha) : \mathcal{K}(a, c) \to \mathcal{K}(a, b)$ defined by $\beta \mapsto \alpha \circ \beta$.

Such a covariant functor is usually denoted $\mathcal{K}(a, -)$, then indeed $F_a(\alpha) = \mathcal{K}(a, \alpha)$.

Then by $F^a : \mathcal{K} \to \mathcal{SET}$; for $b \in \operatorname{Obj}\mathcal{K}$ and $F^a(b) := \mathcal{K}(b, a)$ and for $\alpha : c \to b$, $F^a(\alpha : \mathcal{K}(b, a) \to \mathcal{K}(c, a)$ defined by $\beta \mapsto \beta \circ \alpha$

Likewise, such a covariant functor is usually denoted $\mathcal{K}(-, a)$.

(3) Define $P^+, P^- : \mathcal{SET} \to \mathcal{SET}$. Then for $s \in \operatorname{Obj} \mathcal{SET}$: $P^+(s) = P^-(s) = \mathcal{P}(s)$ and for $\alpha s \to t$, $P^+(\alpha) : \mathcal{P}(s) \to \mathcal{P}(t)$, defined by $\mathcal{P}(s) \ni y \mapsto \alpha[y] = \{\alpha(x) \mid s \in y\}$

$P^-(\alpha) : \mathcal{P}(t) \to \mathcal{P}(s)$ defined by $y \mapsto \alpha^{-1}[y] = \{x \in s \mid \alpha(x) \in y\}$, this is the complete preimage of the set y.

Definition 2.8 (Full, Faithful, One-to-One Functors)**.** Let $F : \mathcal{K} \to \mathcal{H}$ be functor. It is said F is

(1) *full* if for every $a, b \in \operatorname{Obj} \mathcal{K}$, F maps $\mathcal{K}(a, b)$ **onto** $\mathcal{H}(F(a), F(b))$ as given by $\alpha \mapsto F(\alpha)$.

(2) *faithful*, if for each $a, b \in \operatorname{Obj} \mathcal{K}$, F maps $\mathcal{K}(a, b)$ **into** $\mathcal{H}(F(a), F(b))$ as given by $\alpha \mapsto F(\alpha)$. Hence $F(a) = F(b) \Rightarrow a = b$.

(3) *fully faithful* if it is faithful and full.

Remark 2.9. A mnemonic for remembering the term *full* is that the image of the function fills the codomain; a mnemonic for remembering the term *faithful* is that one can have faith $F(a) = F(b)$ implies $a = b$.

Definition 2.10 (Subcategory, Embedding Functor)**.** Let $\mathcal{K}, \mathcal{H}$ be categories. It is said that $\mathcal{K}$ is a *subcategory* of the category $\mathcal{H}$, written $\mathcal{K} \subseteq \mathcal{H}$, if the following conditions are met:

(1) $\operatorname{Obj} \mathcal{K} \subseteq \operatorname{Obj} \mathcal{H}$

(2) $(\forall a, b \in \operatorname{Obj} \mathcal{K})\ \mathcal{K}(a, b) \subseteq \mathcal{H}(a, b)$

(3) $\mathrm{Id} : \mathcal{K} \to \mathcal{H}$, where $\mathrm{Id}(a) = a$ for $a \in \operatorname{Obj} \mathcal{K}$, $\mathrm{Id}(\alpha) = \alpha$ for $\alpha \in \operatorname{Mor} \mathcal{K}$, is a functor, termed an *embedding functor* of $\mathcal{K}$ into $\mathcal{H}$.

Moreover, $\mathcal{K}$ is called a *full subcategory* of $\mathcal{H}$ if Id is full; that is, it inherits all morphisms from $\mathcal{H}$ it can:

$$(\forall a, b \in \operatorname{Obj} \mathcal{K})\, \mathcal{K}(a, b) = \mathcal{H}(a, b).$$

Lemma 2.11 (Functors Preserve Split Morphisms)**.** *Let $F : \mathcal{K} \to \mathcal{H}$, $\alpha \in \mathcal{K}(a, b)$. Then*

(1) If α is a section, then $F(\alpha)$ is a section.

(2) If α is a retraction, then $F(\alpha)$ is a retraction.

(3) If α is an isomorphism, then $F(\alpha)$ is an isomorphism.

Proof. Clear since functors preserve the existence of inverse-mappings between their domains and codomains with respect to individual morphisms.

In greater detail, suppose $\alpha \in \mathcal{K}(a, b)$ is a section with a witness β. It follows from the definition of a functor that

$$\alpha \circ \beta = 1_b \Rightarrow F(\alpha) \circ F(\beta) = 1_{F(b)}$$

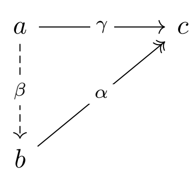
Hence $F(\beta)$ is a witness of $F(\alpha)$ being a section. By analogy, we deduce being a retraction is likewise preserved under functors. Since any isomorphism is both a section and a retraction, it follows trivially the image of an isomorphism under a functor is isomorphic. QED

Remark 2.12. Functors need not preserve monomorphisms nor epimorophisms.

Definition 2.13 (Projective Object)**.** It is said $a \in \operatorname{Obj} \mathcal{K}$ is *projective* in $\mathcal{K}$ if $\mathcal{K}(a, -)$ preserves epimorphisms: given any epimorphism $\alpha \in \mathcal{K}(b, c)$, the following map is surjective

$$\begin{aligned} \mathcal{K}(a, \alpha) : \mathcal{K}(a, b) &\twoheadrightarrow \mathcal{K}(a, c) \\ \beta &\mapsto \alpha \circ \beta \end{aligned}$$

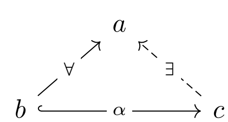
Simply put, given any epimorphism $\alpha \in \mathcal{K}(b, c)$ and any morphism $\gamma \in \mathcal{K}(a, c)$, there exists a morphism $\beta \in \mathcal{K}(a, b)$ such that $\alpha \circ \beta = \gamma$.



Remark 2.14. Definition 2.13 generalises the notion of a projective module. The diagram from Definition Definition 2.13 should be familiar from Module Theory.

Example 2.15.

(1) In $\mathcal{SET}$ all sets are projective objects and all nonempty sets are injective.



(2) In $\mathcal{TOP}$ projective objects are discrete spaces; injective objects are nonempty indiscrete spaces.

(3) In $\mathcal{AB}$ (Abelian groups) projective objects are free groups; that is groups isomorphic to a direct sum over some index-family of any cardinality of cyclic groups.

Injective objects are precisely divisible Abelian groups.

(4) $\mathcal{CHTOP}$ (Complete Hausdorff spaces) projective objects are extremely disconnected, that is those closure of an open set is open, complete and Hausdorff topological spaces.

(5) The category of vectorspaces has only objects are both injective and projective.

(6) $F : \mathcal{AB} \to \mathcal{DAB}$, $G \mapsto$ maximal divisible subgroup is a functor (exercise).

Definition 2.16 (Generator, Cogenerator)**.** $a \in \operatorname{Obj}\mathcal{K}$. It is said that a is a *generator* or a *cogenerator* in $\mathcal{K}$ if $\mathcal{K}(a, -)$ is faithful or $\mathcal{K}(-, a)$ is faithful respectively.

In practice, this means that given any morphism $\alpha \in \mathcal{K}(b, c)$, the assignment $\alpha \mapsto (\beta \mapsto \alpha \circ \beta)$, where $\beta \in \mathcal{K}(a, b)$, is unique.

More simply still, given any two distinct morphisms f, g, there is another morphism h such that $f \circ h \neq g \circ h$, provided the functions are composable. For this reason, generators are sometimes called *separators*.

Example 2.17.

(1) In $\mathcal{SET}$, generators are all nonempty sets and cogenerators are sets with at least two elements.

(2) In $\mathcal{SET}$, an example of a generator is $\mathbb{Z}$ and of a cogenerator is $\mathbb{Q}/\mathbb{Z}$.

(3) In $\mathcal{GRP}$, an example of a generator $\mathbb{Z}$, cogenerators do not exist.

Definition 2.18 (Concrete Category)**.** By a concrete category we understand the pair $(\mathcal{K}, U)$, where $\mathcal{K}$ category $U : \mathcal{K} \to \mathcal{SET}$ is a faithful functor.

Since U is faithful, we may identify each morphism f of $\mathcal{K}$ with the function $U(f)$. In these terms, a concrete category may be described as a category $\mathcal{K}$ in which each object c comes equipped with an 'underlying' set $U(c)$, each morphism $b \to c$ is an actual function $U(b) \to U(c)$, and composition of morphisms is composition of functions.

Definition 2.19 (Mac Lane's Represention)**.** Let $\mathcal{K}$ be small. Its Mac Lane's representation is the functor $M : \mathcal{K} \to \mathcal{SET}$ defined in the following fashion:

$$
\begin{aligned}
& M : \mathcal{K} \to \mathcal{SET} \\
& \operatorname{Obj}\mathcal{K} \to \operatorname{Mor}\mathcal{K} && \operatorname{Mor}\mathcal{K} \to \operatorname{Mor}\mathcal{K} \\
& \quad a \mapsto \bigcup_{b \in \operatorname{Obj}\mathcal{K}} \mathcal{K}(b, a) && \quad \alpha \mapsto (F : M(a) \to M(a') \qquad x \mapsto \alpha \circ x)
\end{aligned}
$$

Theorem 2.20. *Mac Lane's representation functor is injective on $\mathcal{K}$.*

Proof.

(1) Let $a, a' \in \operatorname{Obj}\mathcal{K}$ be distinct. Then $M(a) \cap M(a') = \varnothing$, both of which are nonempty since $1_a \in M(a)$ and $1_{a'} \in M(a')$. Hence it is injective on $\operatorname{Obj}\mathcal{K}$.

(2) Let $\alpha, \beta : a \to a'$ be distinct. Then $M(\alpha)(1_a) = \alpha$ and $M(\beta)(1_a) = \beta$, which implies $M(\alpha) \neq M(\beta)$. Thus, it is faithful.

QED

Corollary 2.20.1. *Every small category $\mathcal{K}$ is concretisable; i.e. there exists a faithful functor $U : \mathcal{K} \to \mathcal{SET}$ such that $(\mathcal{K}, U)$ is concrete.*

Exercise 2.21. Show that there exist categories which are not concretisable; that is, there exists no faithful functor therefrom to $\mathcal{SET}$.

Notation 2.22.

(1) If $\alpha \in \mathcal{K}(a, a')$ is an isomorphism, then $\alpha^{-1} \in \mathcal{K}(a', a)$ denotes a morphism inverse to it. It is uniquely determined (exercise).

(2) Functors may be composed; $F : \mathcal{K} \to \mathcal{H}$, $G : \mathcal{H} \to \mathcal{I}$, then $G \circ F$, or GF, denotes a functor from $\mathcal{K}$ to $\mathcal{I}$ defined by $G \circ F(a) := G(F(a))$.

Definition 2.23 (Category Product)**.** Let $\mathcal{K}, \mathcal{H}$ be categories. Then $\mathcal{K} \times \mathcal{H}$ is defined by

$$\operatorname{Obj}(\mathcal{K} \times \mathcal{H}) = (\operatorname{Obj} \mathcal{K}) \times (\operatorname{Obj} \mathcal{H}) \qquad \operatorname{Mor}(\mathcal{K} \times \mathcal{H}) = (\operatorname{Mor} \mathcal{K}) \times (\operatorname{Mor} \mathcal{H})$$

where all operations are performed componentwise.

Definition 2.24 (Hombifunctor)**.** A functor F is said to be a *hombifunctor* if $F = \mathcal{K}(-,-) : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{SET}$ and

$$\begin{aligned} \mathcal{K}(-,-) : (\operatorname{Obj} \mathcal{K})^2 &\to \operatorname{Mor} \mathcal{K} & \qquad F(\alpha, \beta) : \mathcal{K}(a, b) &\to \mathcal{K}(a', b') \\ F(a, b) &= \mathcal{K}(a, b) & F(\alpha, \beta)(x) &= \beta \circ x \circ \alpha. \end{aligned}$$

$$a' \xrightarrow{\ \alpha\ } a \overset{x}{\dashrightarrow} b \xrightarrow{\ \beta\ } b'$$

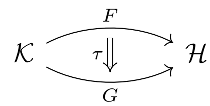
3 Natural Transformations & Yoneda's Lemma

Definition 3.1 (Natural Transformation, Mono-, Epi-, Natural Equivalence)**.** Let $F, G : \mathcal{K} \to \mathcal{H}$ be functors. By a *natural transformation* from F to G we understand the family of morphisms $\tau = \{\tau_a \mid a \in \operatorname{Obj} \mathcal{K}\}$ where each *component* $\tau_a \in \mathcal{H}(F(a), G(a))$ satisfies

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\alpha)} & F(a') \\ {\scriptstyle \tau_a}\downarrow & & \downarrow{\scriptstyle \tau_{\alpha'}} \\ G(a) & \xrightarrow[G(\alpha)]{} & G(a') \end{array}$$

These properties are referred to as *compatibility-conditions.* A natural transformation is said to be a *monotransformation* or an *epitransformation* if all its coponents are mono- or epimorphisms respectively. By a *natural equivalence* or sometimes *natural isomorphism* we understand such a natural transformation whose components are isomorphisms.

Observe that if τ is a natural equivalence, then $\tau^{-1} = \left\{ (\tau_a)^{-1} \,\middle|\, a \in \operatorname{Obj}\mathcal{K} \right\}$ is likewise a natural transformation. In diagrams, we indicate $\mathcal{K}, \mathcal{H}$ having natural transformation τ thus



Example 3.2. Let $a, c \in \operatorname{Obj}\mathcal{K}$, $\alpha \in \mathcal{K}(a, c)$, and consider the following covariant homfunctors $\mathcal{K}(a, -), \mathcal{K}(c, -) : \mathcal{K} \to \mathcal{SET}$.

Let $\alpha \in \mathcal{K}(a, c)$ and set $\tau := \mathcal{K}(\alpha, -) : \mathcal{K}(c, -) \to \mathcal{K}(a, -)$, a contravariant homfunctor, and for each $b \in \operatorname{Obj}\mathcal{K}$ put

$$\begin{aligned} \tau_b = \mathcal{K}(\alpha, b) : \mathcal{K}(c, b) &\to \mathcal{K}(a, b) \\ \tau_b : x &\mapsto x \circ \alpha. \end{aligned}$$

We need to check τ satisfies the compatibility-conditions. Let $b \in \operatorname{Obj}\mathcal{K}$ and $\beta : b \to b'$. Then

$$\begin{array}{ccc} \mathcal{K}(c, b) & \xrightarrow{\tau_b} & \mathcal{K}(a, b) \\ {\scriptstyle \mathcal{K}(c,\beta)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{K}(a,\beta)} \\ \mathcal{K}(c, b') & \xrightarrow{\tau_{b'}} & \mathcal{K}(a, b') \end{array}$$

as for any $x \in \mathcal{K}(c, b)$

$$\left[\mathcal{K}(a, \beta) \circ \tau_b\right](x) = \beta \circ x \circ \alpha = \left[\tau_{b'} \circ \mathcal{K}(c, \beta)\right](x).$$

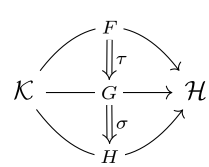
Definition 3.3 (Conglomerate)**.** By a *conglomerate*, we understand a collection of classes. This is an informal term that, though it may be formalised, is to be understood in the intuitive sense.

Notation 3.4. Let $F, G : \mathcal{K} \to \mathcal{H}$ be functors. By $\operatorname{Nat}(F, G)$ we understand the conglomerate of all natural transformations from F to G.

Moreover, $\mathcal{H}^{\mathcal{K}}$ denotes the conglomerate of all covariant functors from $\mathcal{K}$ to $\mathcal{H}$. It itself said to be a *quasicategory*, whose objects are functors and whose morphisms are natural transformations. It behaves exactly like a category but is not one due to the unbounded size of its collections of objects and morphisms. Its identity (natural transformation) is given by $\operatorname{Id} : F \to F$, $\operatorname{Id}_a = 1_{F(a)}$

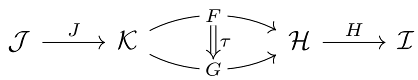
Remark 3.5.

(1) Natural transformations may be composed. Let $\tau : F \to G$ and $\sigma : G \to H$, then $\sigma \circ \tau : F \to H$ where $(\sigma \circ \tau)_a := \sigma_a \circ \tau_a \in \mathcal{H}(F(a), H(a))$.



(2) $H\tau : HF \to HG$, $(H\tau)_a \in \mathcal{I}(HF(a), HG(a))$, $a := H(\tau_a)$.

$\tau J : FJ \to GJ$, $(\tau J) \in \mathcal{H}(FJ(c), GJ(c), GJ(c))$, where $c := \tau_{J(c)}$.

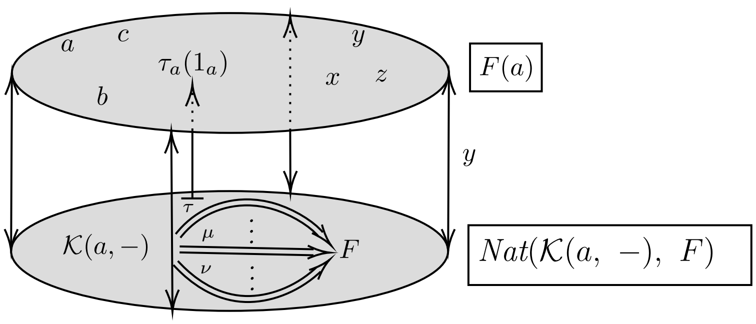


3.1 Yoneda's Lemma

Yoneda's Lemma is arguably the most important result in category theory. It is an abstract result on functors of the type *morphisms into a fixed object*; a vast generalisation of Cayley's Theorem from Group Theory (viewing a group as a miniature category with just one object and only isomorphisms). It allows the embedding of any locally small category into a category of contravariant functors defined thereon. It also clarifies how the embedded category of representable functors and their natural transformations relates to the other objects in the larger functor-category. It is named after a Japanese mathematician Nobuo Yoneda.

Lemma 3.6 (Yoneda I)**.** *Let $\mathcal{K}$ be a category, $a \in \operatorname{Obj}\mathcal{K}$ and $F : \mathcal{K} \to \mathcal{SET}$. Then the following map is bijective.*

$$y : \operatorname{Nat}(\mathcal{K}(a, -), F) \to F(a) \qquad \tau \mapsto \tau_a(1_a) \in F(a).$$



Proof. Set for each $b \in \operatorname{Obj}\mathcal{K}$,

$$z : F(a) \to \operatorname{Nat}(\mathcal{K}(a, -), F) \qquad \tau_b^x : \mathcal{K}(a, b) \to F(b)$$
$$x \mapsto \tau^x \qquad \alpha \mapsto [F(\alpha)](x)$$

$$\begin{array}{ccc} \mathcal{K}(a, b) & \xrightarrow{\mathcal{K}(a,\beta)} & \mathcal{K}(a, b') \\ \Big\downarrow {\scriptstyle \tau_b^x} & & \Big\downarrow {\scriptstyle \tau_{b'}^x} \\ F(b) & \xrightarrow{F(\beta)} & F(b') \end{array}$$

To verify z is well-defined, we need to check τ^x satisfies compatibility-conditions. Put $\alpha \in \mathcal{K}(a, b)$, then

$\circlearrowleft$

$$[F(\beta)\tau_b^x](\alpha) = [F(\beta) \circ F(\alpha)](x) = F(\beta \circ \alpha)(x)$$

$\circlearrowright$

$$[\tau_{b'}^x \circ \mathcal{K}(a, \beta)](\alpha) = \tau_{b'}^x(\beta \circ \alpha) = F(\beta \circ \alpha)(x)$$

We shall now check in two steps that z is the inverse map to y and vice versa, thus proving the claim.

$z \circ y = \mathrm{Id}_{\mathrm{Nat}(\mathcal{K}(a,-),F)}$ We wish to verify $z(y(\tau)) = z(\tau_a(1_a)) = \tau^{\tau_a}(1_a) = \tau$. Let $b \in \mathrm{Obj}$ and $\alpha \in \mathcal{K}(a, b)$. Then

$$\mathcal{K} : \tau_b^{\tau_a(1_a)}(\alpha) = [F(\alpha)](\tau_a(1_a)) = \tau_b \circ \underbrace{\mathcal{K}(a, \alpha)(1_a)}_{\alpha} = \tau_b(\alpha).$$

$$\begin{array}{ccc} \mathcal{K}(a,a) & \xrightarrow{\tau_a} & F(a) \\ {\scriptstyle \mathcal{K}(a,\alpha)}\downarrow & & \downarrow{\scriptstyle F(\alpha)} \\ \mathcal{K}(a,b') & \xrightarrow{\tau_{b'}} & F(b') \end{array}$$

$y \circ z = \mathrm{Id}_{F(a)}$ Let $x \in F(a)$. Then

$$y \in z = \mathrm{Id}_{F(a)} : y(z(x)) = y(\tau^x) = \tau_a^x(1_a) = [F(1_a)](x) = 1_{F(a)}(x) = x.$$

QED

Remark 3.7. A natural transformation is, formally speaking, a formula defining a class of morphisms.

Remark 3.8. In Lemma 3.6, we fixed an object a and a functor F. We shall investigate what would happen if we allowed these be variable. It follows then we need to introduce new notation: we donote the a and F whereupon y from Lemma 3.6 depends by writing $y_{a,F}$.

Lemma 3.9 (Yondeda II)**.** *Let* $y = \left\{ y_{a,F} \,\middle|\, a \in \mathrm{Obj}\,\mathcal{K} \wedge F \in \mathrm{Obj}\,\mathcal{SET}^{\mathcal{K}} \right\}$, *a natural equivalence of the functors* $N, E : \mathcal{K} \times \mathcal{SET}^{\mathcal{K}} \to \mathcal{SET}$ *defined thus*

$$\begin{array}{lll} N(a,F) := \mathrm{Nat}(\mathcal{K}(a,-),F) & \alpha \in \mathcal{K}(a,b) & N(\alpha,\rho) : N(a,F) \to N(b,G) \\ N(b,G) := \mathrm{Nat}(\mathcal{K}(b,-),G) & \rho \in \mathrm{Nat}(F,G) & [N(\alpha,\rho)](\nu) := \rho \circ \nu \circ \mathcal{K}(\alpha,-) \end{array}$$

$$\begin{array}{ccc} \mathcal{K}(a,-) & \xrightarrow{\quad \nu \quad} & F \\ {\scriptstyle \mathcal{K}(\alpha,-)}\uparrow & & \downarrow{\scriptstyle \rho} \\ \mathcal{K}(b,-) & & G \end{array}$$

$$E(a, F) := F(a) \quad E(b, G) := G(b) \quad E(\alpha, \rho) := \rho_b \circ F(\alpha) \quad = G(\alpha \circ \rho_a)$$

Proof. The following proof shall not be examined. From theorem [3.6], we already know $y_{a,F}$ is a bijection, and hence an isomorphism in $\mathcal{SET}$, for every $a \in \operatorname{Obj}\mathcal{K}$, $F : \mathcal{K} \to \mathcal{SET}$. Hence, it only remains to show it satisfied the compatibility-conditions.

Let $\tau \in N(a, F)$. Then $E(\alpha, \rho)(y_{a,F}(\tau)) = E(\alpha, \rho)(\tau_a(1_a)) = \rho_b \circ \tau_b \circ \mathcal{K}(a, \alpha)(1_a) = [\rho_b \circ \tau_b](\alpha)$.

To see the lower triangle in the following diagram is correct, write:

$$y_{b,G}(N * (\alpha, \rho)(\tau)) = y_{b,G}(\underbrace{\rho \circ \tau \circ \mathcal{K}(\alpha, -)}_{\in N(b,G)}) = (\rho \circ \tau \circ \mathcal{K}(\alpha, -))_b(1_b) = [\rho_b \circ \tau_b \circ \underbrace{\mathcal{K}(\alpha, b)}_{= \mathcal{K}(\alpha, -)_b}](1_b) = [\rho_b \circ \tau_b](\alpha)$$

$$\begin{array}{ccc}
N(a, F) & \xrightarrow{\quad y_{a,F} \quad} & E(a, F) = F(a) \\
\Big\downarrow {\scriptstyle N(\alpha,\rho)} & \mathcal{SET} & \Big\downarrow {\scriptstyle E(\alpha,\rho)} \\
N(b, G) & \xrightarrow{\quad y_{b,G} \quad} & E(b, G) = G(b)
\end{array}$$

QED

Remark 3.10. In practice, Yoneda's lemma allows us, given a formula defining a natural transformation and a set, to uniquely assign an element of the set; it gives us a twofold way of doing this constructively.

Definition 3.11 (Yoneda's Embedding)**.** Let $\mathcal{K}$ be a category. Then *Yondeda's embedding* is a functor $Y : \mathcal{K}^{\text{op}} \to \mathcal{SET}^{\mathcal{K}}$ defined by $a \in \operatorname{Obj}\mathcal{K}$, $\alpha \in \mathcal{K}(b, a)$,

$$Y : a \mapsto \mathcal{K}(a, -) \qquad \alpha \mapsto \mathcal{K}(\alpha, -) : \mathcal{K}(a, -) \to \mathcal{K}(b, -).$$

Remark 3.12. Sometimes, Yondeda's embedding is defined as $\tilde{Y} : \mathcal{K} \to \mathcal{SET}^{\mathcal{K}^{\text{op}}}$ by $\tilde{Y}(a) := \mathcal{K}(-, a)$. This is done so because $\mathcal{SET}^{\mathcal{K}^{\text{op}}}$ is rather important; it is called the *quasicategory of presheaves.*

Remark 3.13. Yoneda's embedding is a vast generalisation of Cayley's theorem from Group Theory

$$G \hookrightarrow S(G).$$

Theorem 3.14 (Yoneda's Embedding Is an Embedding)**.** *Yoneda's Embedding Y is a full and injective.*

Proof.

(1) Y is injective on objects, since given any two distinct a, b, $\mathcal{K}(a, a) \cap (\mathcal{K}(b, a) = \varnothing$, the former of which is nonempty since $1_a \in \mathcal{K}(a, a)$. Recall

$$Y(a) = \mathcal{K}(a, -) \qquad\qquad Y(b) = \mathcal{K}(b, -)$$

(2) Y is faithful, since given any $\beta \neq \alpha \in \mathcal{K}(b, a)$

$$\begin{aligned} Y(\alpha) &= \mathcal{K}(\alpha, -) \qquad\qquad & [Y(\alpha)]_a(1_\alpha) &= \alpha \\ Y(\beta) &= \mathcal{K}(\beta, -) \qquad\qquad & [Y(\beta)]_a(1_\beta) &= \beta \end{aligned}$$

because $[\mathcal{K}(\alpha, -)]_a = \mathcal{K}(\alpha, a)$ and $\mathcal{K}(\alpha, a)(1_a) = 1_a \circ \alpha = \alpha$ and likewise for β. Hence $Y(\alpha) \neq Y(\beta)$.

(3) Y is full. We shall show that for any $\tau : \mathcal{K}(a, -) \to \mathcal{K}(b, -)$ there is a suitable $\alpha \in \mathcal{K}(b, a)$ such that $\tau = \mathcal{K}(\alpha, -)$. From the proof of Theorem 3.6, take $\tau = \tau^\alpha$ for some $\alpha \in F(a) = \mathcal{K}(b, a)$.

$$\tau_c^\alpha(\beta) = [F(\beta)](\alpha) = [\mathcal{K}(b, \beta)](\alpha) = \beta \circ \alpha,$$

where $\beta \in \mathcal{K}(a, c)$.

Note

$$\begin{aligned} [\mathcal{K}(\alpha, -)]_c : \mathcal{K}(a, c) &\to \mathcal{K}(b, c) \\ b &\mapsto \beta \circ \alpha \end{aligned}$$

QED

Definition 3.15 (Universal Pair)**.** Let $F : \mathcal{K} \to \mathcal{SET}$ be a functor. Then (a, x), where $a \in \operatorname{Obj} \mathcal{K}$ and $x \in F(a)$ is said to be a *universal pair*[2] of the functor F if for each (b, y) where $b \in \operatorname{Obj} \mathcal{K}$, $y \in F(b)$ there exists a unique $\alpha \in \mathcal{K}(a, b)$ such that $[F(\alpha)](x) = y$.

$$\begin{array}{ccc} a & \overset{\alpha}{\dashrightarrow} & b \\ & \Vert & \\ & F & \\ & \Downarrow & \\ x & \overset{F(\alpha)}{\dashrightarrow} & y \end{array}$$

Theorem 3.16 (Representable Functor Equivalent Conditions)**.** *Let $F : \mathcal{K} \to \mathcal{SET}$ be a functor. Then the following are equivalent:*

(1) F has a universal pair.

(2) There exists $a \in \operatorname{Obj} \mathcal{K}$ such that $\mathcal{K}(a, -)$ is naturally equivalent with F.

[2]The term used on *Wikipedia* and *Categories for the Working Mathematician* is *universal element.*

Proof. (**1**) ⇒ (**2**) Let (a, x) be a universal pair of $F : \mathcal{K} \to \mathcal{SET}$. Set for each $b \in \operatorname{Obj}\mathcal{K}$,

$$\begin{aligned} z : F(a) &\to \operatorname{Nat}(\mathcal{K}(a,-), F) \\ x &\mapsto \tau^x \end{aligned} \qquad\qquad \begin{aligned} \tau^x_b : \mathcal{K}(a,b) &\to F(b) \\ \alpha &\mapsto [F(\alpha)](x) \end{aligned}$$

as we have in the proof of Lemma 3.6, whence we know it is a well-defined natural transformation. We will show $\tau^x : \mathcal{K}(a,-) \to F$ is the desired natural equivalence. To this end, we need to prove τ^x_b is an isomorphism for each $b \in \operatorname{Obj}\mathcal{K}$. The target-category of $F, \mathcal{K}(a,-)$ is $\mathcal{SET}$ and thus from Example 1.21, we know this occurs iff τ^x is bijective.

Let $b \in \operatorname{Obj}\mathcal{K}$ and $v \in F(b)$. Then

$$\begin{array}{ccc} 1_a \in \mathcal{K}(a,a) & \xrightarrow{\tau^x_a} & F(a) \ni x \\ \downarrow & & \downarrow \\ \mathcal{K}(a,b) & \xrightarrow{\tau^x_b} & F(b) \ni v \end{array} \qquad\qquad \begin{array}{ccc} 1_a & \longmapsto & x \\ \downarrow & & \downarrow \\ \alpha & \longmapsto & v \end{array}$$

Note $\tau^x_a(1_a) = [F(1_a)](x) = x$. By virtue of (a, x) being a universal pair, there exists a unique $\alpha \in \mathcal{K}(a,b)$ such that $F(\alpha)(x) = v$. Rephrased, $v \in F(b)$ has a unique preimage $\alpha \in \mathcal{K}(a,b)$ under τ^x_b. In either case, τ^x_b is bijective.

(**2**) ⇒ (**1**)**.** Recall the following bijection from Lemma 3.6.

$$y : \operatorname{Nat}(\mathcal{K}(a,-), F) \to F(a) \qquad\qquad \tau \mapsto \tau_a(1_a) \in F(a)$$

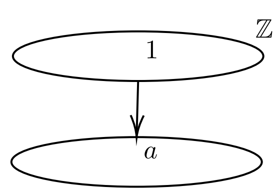
and let $\tau : \mathcal{K}(a,-) \to F$ be a natural equivalence. Then $(a, y(\tau)) = (a, \tau_a(1_a))$ is a universal pair. To see this, let $(b, v) \in \operatorname{Obj}\mathcal{K} \times \operatorname{Obj}\mathcal{SET}$. Then the desired unique morphism is $\alpha = \tau_b^{-1}(v)$. QED

Definition 3.17 (Representable Functor)**.** A functor $F : \mathcal{K} \to \mathcal{SET}$ is said to be *representable* if it has a universal pair.

Example 3.18.

(1) Consider the forgetful functor $U : \mathcal{TOP} \to \mathcal{SET}$. Its universal pair is $(\{*\}, *)$.

(2) Consider the forgetful functor $U : \mathcal{GRP} \to \mathcal{SET}$. Its universal pair is $((\mathbb{Z}, +, -, 0), 1)$. Observe $U(-) \simeq \mathcal{GRP}((\mathbb{Z}, +, -, 0), -)$.



(3) Let $U : \mathrm{Rng} \to \mathcal{SET}$ be a forgetful functor. Its universal pair is $(\mathbb{Z}[x], x)$

Definition 3.19 (Essentially Surjective Functor)**.** Let $F : \mathcal{K} \to \mathcal{H}$ be a functor. It is said F is *essentially surjective* or *dense* if for each $b \in \mathrm{Obj}\,\mathcal{H}$ there exists $a \in \mathrm{Obj}\,\mathcal{K}$ such that $F(a) \simeq b$.

Theorem 3.20 (Split Image Implies Split Preimage under Functors)**.** *Let $F : \mathcal{K} \to \mathcal{H}$ is faithful and full. Then whenever $F(\alpha)$ is a split morphism for some $\alpha \in \mathrm{Mor}\,\mathcal{K}$, then α is a like split morphism.*

Proof. Let $\alpha \in \mathcal{K}(a, b)$. Suppose $F(\alpha)$ is a section; it is the right inverse of some $\beta \in \mathcal{H}(F(b), F(a))$ — $\beta \circ F(\alpha) = 1_{F(a)}$. Since F is full, there is some $\gamma \in \mathcal{K}(b, a)$ such that $F(\gamma) = \beta$,

$$F(1_a) = 1_{F(a)} = F(\gamma) \circ F(\alpha) = F(\gamma \circ \alpha)$$

and since F is also faithful, we have $1_a = \gamma \circ \alpha$ whence α is a section. The dual statement about retractions follows by the symmetry of our argument. QED

Remark 3.21. Theorem 3.20 is a complementary statement to Lemma 2.11.

Definition 3.22 (Skeleton)**.** Let $\mathcal{K}$ be a category and $S \subseteq \mathrm{Obj}\,\mathcal{K}$. It is said S is a skeleton of $\mathcal{K}$ if

(1) $(\forall a, b \in S)(a \neq b \to a \not\simeq b)$.

(2) $(\forall a \in \mathrm{Obj}\,\mathcal{K})(\exists b \in S)(a \simeq b)$.

Example 3.23. In $\mathcal{SET}$, the class of all cardinal numbers is a skeleton.

Remark 3.24. The proposition 'every category has a skeleton' is equivalent to the *Axiom of Global Choice*. This axiom if consistent with ZFC and Goedel-Bernay's Axiomatisation.

Remark 3.25. If $F : \mathcal{K} \to \mathcal{H}$ is a full, faithful, and essentially surjective functor, then for any skeleton $S \subseteq \mathrm{Obj}\,\mathcal{K}$, the class $F(S) = \{F(s), | \, s \in S\}$ is a skeleton of the category $\mathcal{H}$.

Proof. The first condition in the definition of a skeleton follows from Theorem 3.20 and the second from F being essentially surjective. QED

Definition 3.26 (Category Equivalence)**.** Let $\mathcal{K}$, $\mathcal{H}$ be categories. It is said $\mathcal{K}$ is equivalent with $\mathcal{H}$, written $\mathcal{K} \simeq \mathcal{H}$ if there exist functors $F : \mathcal{K} \to \mathcal{H}$ and $G : \mathcal{H} \to \mathcal{K}$ such that $G \circ F \simeq \mathrm{Id}_{\mathcal{K}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{H}}$.[3]

Remark 3.27. We have now two notions of '$\simeq$'. One sense is the one we have just now defined, and the second as isomorphism between two objects of a category. If we considered then a category of categories, these two notions would merge.

[3] '$\simeq$' denotes a natural equivalence.

Theorem 3.28 (Category Equivalence)**.** *Let $\mathcal{K}, \mathcal{H}$ be categories. Then $\mathcal{K} \simeq \mathcal{H}$ iff there exists $F : \mathcal{K} \to \mathcal{H}$ which is faithful, full and essentially surjective.*

Proof. ($\Leftarrow$) We shall need to assume the existenece of a skeleton S. Assume then S is a skeleton of $\operatorname{Obj}\mathcal{K}$. Let $F : \mathcal{K} \to \mathcal{H}$ be faithful, full and essentially surjective, by Theorem (above supershort proof). For each $b \in \operatorname{Obj}\mathcal{H}$, let $s_b \in S$ be the only objects for which $F(s_b) \simeq b$. Fix an isomorphism $\iota_b : b \to F(s_b)$. Define $G : \mathcal{H} \to \mathcal{K}$ as follows

(1) For $b \in \operatorname{Obj}\mathcal{H}$ we put $G(b) = s_b$.

(2) For $\beta \in \mathcal{H}(b, c)$ let $G(\beta) : s_b \to s_c$ be the only one such that $F \circ G(\beta) = \iota_c \circ \beta \circ \iota_b^{-1}$.

$$\begin{array}{ccc} F(s_b) & \xleftarrow{\ \iota_b\ } & b \\ & & \Big\downarrow{\scriptstyle \beta} \\ F(s_c) & \xleftarrow{\ \iota_c\ } & c \end{array}$$

Is G a functor? $G(1_b)$ is the only one such that $F(G(1_b)) = \iota_b \circ 1_b \circ \iota_b^{-1} = 1_b$ and therefore $G(1_b) = 1_{G(b)} = 1_{s_b}$.

Likewise $G(\gamma \circ \beta) = G(\gamma) \circ G(\beta)$ iff $F \circ G(\gamma \circ \beta) = F(G(\gamma) \circ G(\beta)) = F \circ G(\gamma) F \circ G(\beta)$.

$\iota_b : b \to F \circ G(b)$ and $\iota := \{\iota_b \,|\, b \in \operatorname{Obj}\mathcal{H}\}$. We claim that ι is a natural rquivalence. $\iota : \mathrm{Id}_{\mathcal{H}} \to F \circ G$. We know ι_b are isomorphisms for each $b \in \operatorname{Obj}\mathcal{H}$. It remains to verify the natural transformation conditions.

$$\begin{array}{ccc} b & \xrightarrow{\ \iota_b\ } & F \circ G(b) \\ {\scriptstyle \beta}\Big\downarrow & & \Big\downarrow{\scriptstyle F \circ G(\beta) = \iota_c \circ \beta \iota_b{}^{-1}} \\ c & \xrightarrow{\ \iota_c\ } & F \circ G(c) \end{array}$$

Upper Triangle: $\iota_c \circ \beta \circ \iota_b^{-1} \circ \iota_b = \iota_c \circ \beta$. Lower Triangle: $\iota_c \circ \beta$.

It remains to define natural equivalence $\tau : \mathrm{Id}_{\kappa} \to G \circ F$. The map τ_a is defined as the only morphism from $\mathcal{K}(a, GF(a))$ for which $F(\tau_a) = \iota_{F(a)} : F(a) \to F(s_{f(a)}) = F(G \circ F(a))$; $\tau_a := F^{-1}(\iota_{F(a)})$.

$$\begin{array}{ccc} a & \xrightarrow{\ \tau_a\ } & G \circ F(a) = s_{F(a)} \\ {\scriptstyle \alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle G \circ F(\alpha)} \\ a' & \xrightarrow{\ \tau_{a'}\ } & G \circ F(a') = s_{F(a')} \end{array}$$

The fact this diagram commutes shall be proven by applying F and showing this new square commutes, this works because F is full and faithful.

$$\begin{array}{ccc} F(a) & \xrightarrow{\iota_{F(a)}} & F\left(s_{F(a)}\right) \\ \Big\downarrow{\scriptstyle F(\alpha)} & & \Big\downarrow{\scriptstyle F\circ G(F(\alpha))} \\ F(a') & \xrightarrow{\iota_{F(a')}} & F\left(s_{F(a')}\right) \end{array}$$

Which commutes since ι is a natural transformation.

($\Rightarrow$) Let $F : \mathcal{K} \to \mathcal{H}$, $G : \mathcal{H} \to \mathcal{K}$ satisfy the conditions from the definition. of $\mathcal{K} \simeq \mathcal{H}$. We need to show F is full, faithful and essentially surjective.

Faithful.

$$\begin{array}{ccc} a & \xrightarrow{\tau_a} & G \circ F(a) \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle G\circ F(\alpha)=\tau_{a'}\circ\alpha\circ\tau_a^{-1}} \\ a' & \xrightarrow{\tau_{a'}} & G \circ F(a') \end{array}$$

$\tau : \mathrm{Id}_{\mathcal{K}} \to G \circ F$ is a natural equivalence. Recall then $\tau_a, \tau_{a'}$ are isomorphisms. Then $G \circ F(\alpha) = \tau_{a'} \circ \alpha \circ \tau_a^{-1}$ and $\tau_{a'} \circ \beta \circ \tau_a^{-1}$ implies $G \circ F(\alpha) \neq G \circ F(\beta) \Rightarrow F(\alpha) \neq F(\beta)$.

Esssentially Surjective. $\iota : \mathrm{Id}_{\mathcal{H}} \to F \circ G$ is a natural equivalence. $(\forall b \in \mathrm{Obj}\,\mathcal{H}) : b \to F(G(b))$ mapped by ι_b is an isomorphism.

By symmetry, G is faithful and G is essentially surjective.

Full.

$$\begin{array}{ccc} a & \xrightarrow{\tau_a} & G \circ F(a) \\ \Big\downarrow{\scriptstyle \tau_a} & & \Big\downarrow{\scriptstyle G\circ F(\tau_a)} \\ G \circ F(a) & \xrightarrow{\tau_{G\circ F(a)}} & G \circ F \circ G \circ F(a) \end{array}$$

$G \circ F(\tau_a) \circ \tau_a = \tau_{G \circ F(a)} \circ \tau_a$. This implies $G \circ F(\tau_a) = \tau_{G\circ F(a)}$.

Let $a, a' \in \mathrm{Obj}\,\mathcal{K}$, $\gamma \in \mathcal{H}(F(a), F(a'))$. We are looking for $\alpha \in \mathcal{K}(a, a')$ such that $F(\alpha) = \gamma$. Set $\alpha = \tau_{a'}^{-1} \circ G(\gamma) \circ \tau_a$. It suffices to show $G \circ F(\alpha) = G(\gamma)$. Since G is faithful, this implies $F(\alpha) = \gamma$.

$$\begin{array}{ccc} a & \xrightarrow{\tau_a} & G\circ F(a) \\ \Big\downarrow{\scriptstyle \alpha?} & & \Big\downarrow{\scriptstyle G(\gamma)} \\ a' & \xrightarrow{\tau_{a'}} & G\circ F(a') \end{array}$$

$G\circ F(\alpha) = G\circ F(\tau_{a'}^{-1}\circ G(\gamma)\circ \tau_a) = G\circ F(\tau_{a'}^{-1})\circ GFG(\gamma\circ GF(\tau_a) = \tau_{GF(a')}^{-1}\circ GFG(\gamma)\circ \tau_{GF(a)} = G(\gamma)$.

$$\begin{array}{ccc} G\circ F(a) & \xrightarrow{\tau_{G\circ F(a)}} & G\circ F\circ G\circ F(a) \\ \Big\downarrow{\scriptstyle G(\gamma)} & & \Big\downarrow{\scriptstyle G\circ F\circ G(\gamma)} \\ G\circ F(a') & \xrightarrow{\tau_{G\circ F(a')}} & G\circ F\circ G\circ F(a') \end{array}$$

QED

Example 3.29. Is there a set equivalent with $\mathcal{SET}^{\text{op}}$? We have already encountered the contravariant functor $P^- : \mathcal{SET}\to\mathcal{SET}$ in Example 2.7. Let us view it as $P^- : \mathcal{SET}^{\text{op}} \to \mathcal{SET}$. Recall $P^-(a) = \mathcal{P}(a)$ and let $\alpha \in \mathcal{SET}(a,b)$ and $x \subseteq b$. Then

$$\begin{aligned} P^-(\alpha) : \mathcal{P}(b) &\to \mathcal{P}(a) \\ x &\mapsto \alpha^{-1}(x) \subseteq a. \end{aligned}$$

Let $\mathcal{P}$ be a complete subcategory of $\mathcal{SET}$ whose objects are of the form $\mathcal{P}(x)$ for each $x \in \mathcal{SET}$.

Then P^- is faithful (even invertible) and essentially surjective on $\mathcal{P}$. The functor $P^- : \mathcal{SET}^{\text{op}} \to \mathcal{P}$, however, is not full. To correct this, we restrict $\mathcal{P}$ only to morphisms preserving unions, intersections, the emptyset, and complements; if we take the subcategory $\mathcal{B} \subseteq \mathcal{P}$, $\operatorname{Obj}\mathcal{B} = \operatorname{Obj}\mathcal{P}$ with fewer morphisms, then $P^- : \mathcal{SET}^{\text{op}} \to \mathcal{B}$ witnesses $\mathcal{SET}^{\text{op}} \simeq \mathcal{B}$.

4 Limits & Colimits

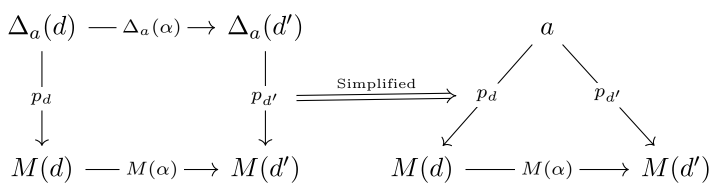
Notation 4.1. For categories $\mathcal{K}, \mathcal{H}$, and object $a \in \operatorname{Obj} \mathcal{H}$, we define $\Delta_a : \mathcal{K} \to \mathcal{H}$, a *constant functor* onto a by

$$\Delta_a(b) = a \text{ for all } b \in \operatorname{Obj} \mathcal{K}\, \Delta_a(\beta) = 1_a \text{ for all } \beta \in \operatorname{Mor} \mathcal{K}\,.$$

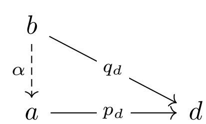
Definition 4.2 (Diagram, Scheme, Cone, Limit)**.**

(1) By a *diagram* in the category $\mathcal{K}$, we understand the functor $M : \mathcal{D} \to \mathcal{K}$, where $\mathcal{D}$ is a small category; $\mathcal{D}$ is termed the *diagram scheme* of M

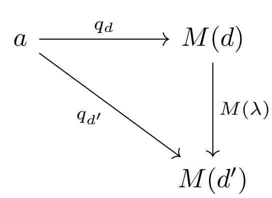
(2) By the *cone* of diagram $M : \mathcal{D} \to \mathcal{K}$ we understand the pair (a, p), where $a \in \operatorname{Obj} \mathcal{K}$ and $p \in \operatorname{Nat}(\Delta_a : \mathcal{D} \to \mathcal{K},\ M)$. In this context, a is termed the *apex* of the cone (a, p).



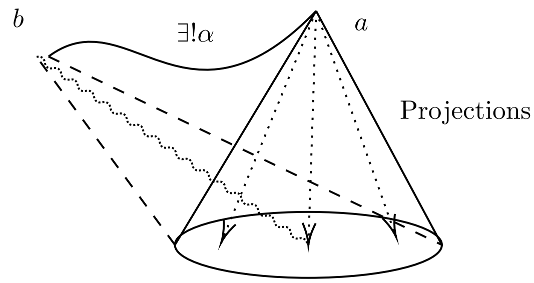
(3) A cone (a, p) of a diagram $M : \mathcal{D} \to \mathcal{K}$ is termed the *limit* of the diagram M if for every cone (b, q) of the diagram M, there exists a unique $\alpha \in \mathcal{K}(b, a)$ such that $(\forall d \in \operatorname{Obj} \mathcal{D})\, q_d = p_d \circ \alpha$. We write $(a, p) = \lim M$. The components $p_d : a \to M(d)$ are termed *limit projections.*



Remark 4.3 (Properties & Further Terminology)**.** What does it mean for (b, q) to be a cone? For all $d, d' \in \operatorname{Obj} D$ and $(\forall \lambda \in D(d, d'))M \circ q_d = q_{d'}$.



The term *cone* was motivated by the following image



Exercise 4.4. If $(a, p), (b, q)$ are limits of diagram M, then a, b are isomorphic.

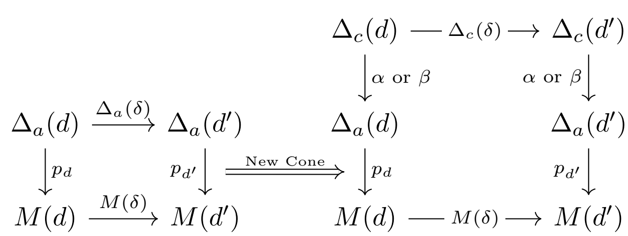
Solution. By definition, there exists $\alpha \in \mathcal{K}(b, a)$ and thus $\beta \in \mathcal{K}(a, b)$ such that $(\forall d \in \operatorname{Obj} D)\, q_d = p_d \circ \alpha$ and $p_d = q_d \circ \beta$. Then $p_d = p_d \circ (\alpha \circ \beta)$ for all d whence $\alpha \circ \beta = 1_a$ since $p_d = p_d \circ 1_a$ holds and 1_a is unique by definition of a limit. Proceed by analogy for $\beta \circ \alpha = 1_b$. QEF

Theorem 4.5 (Morphisms Are Determined by Cone-Projections)**.** *Let (a, p) be a limit of the diagram M and $\alpha, \beta \in \mathcal{K}(c, a)$. Then $\alpha = \beta$ iff $(\forall d \in \operatorname{Obj} D)\, p_d \circ \alpha = p_d \circ \beta$.*

Proof. $(\Rightarrow)$ Trivially. $(\Leftarrow)$ Consider the cone

$$(c, \{p_d \circ \alpha \mid d \in \operatorname{Obj} D\}) = (c, \{p_d \circ \beta \mid d \in \operatorname{Obj} D\})$$

of M. Observe it indeed is a cone: the morphisms $p_d \circ \alpha$ and $p_d \circ \beta$ are components of a natural transformation as evidenced by the following commutative diagram for $\delta \in D(d, d')$:



By definition of a limit, there exists a *unique* $\gamma \in \mathcal{K}(c, a)$ such that $(\forall d \in \operatorname{Obj} D)\, p_d \circ \alpha = p_d \circ \gamma$ and $p_d \circ \beta = p_d \circ \gamma$ whence $\alpha = \beta = \gamma$. QED

Remark 4.6. The dual notions of cone and limit are *cocone* and *colimit* which are defined by analogy.

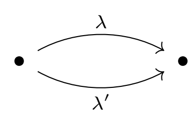
In greater detail, the cocone of the diagram $M : D \to \mathcal{K}$ is the pair (a, i) where $a \in \operatorname{Obj} \mathcal{K}$, and i is a natural transformation from M to Δ_a. In the definition of a colimit α maps the apex of a colimit cone to the apex of a general cocone. i_d is a colimit injection.

Definition 4.7 (Terminal & Initial Objects)**.** The (apex of the) limit of a diagram with an empty schema is termed a *terminal object* of the category $\mathcal{K}$. In practice, an object t is terminal if for every $a \in \operatorname{Obj}\mathcal{K}$ there exists a unique morphism $\alpha \in \mathcal{K}(a,t)$.

Dually, one defines the *initial object* as the colimit of a diagram with a nonempty schema. In practice, an object i is initial if for every $b \in \operatorname{Obj}\mathcal{K}$, there exists a unique $\alpha \in \mathcal{K}(i,b)$.

Lastly, a *null object* is an object which is both initial and terminal, denoted 0. We define a *zero morphism* from α to β by $\alpha \overset{\exists!}{\to} 0 \overset{\exists!}{\to} \beta$.

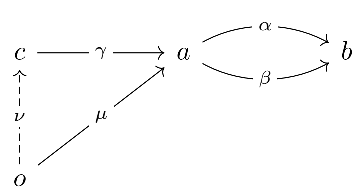
Definition 4.8 (Equalisation & Equaliser)**.** Let $\alpha, \beta \in \mathcal{K}(a,b)$ and take the scheme $\mathcal{D}$ defined by the diagram



We say that $\gamma \in \mathcal{K}(c,a)$ *equalises* α and β if $\alpha \circ \gamma = \beta \circ \gamma$. This is equivalent to saying $(c, \{\gamma, \alpha \circ \gamma\})$ is the cone of a diagram $M : D \to \mathcal{K}$ such that $M(x) = a, M(y) = b, M(\lambda) = \alpha, M(\lambda') = \beta$.

By the *equaliser* of α and β, written $\operatorname{eq}(\alpha,\beta)$ we understand $\gamma \in \mathcal{K}(c,a)$ such that $(c, \{\gamma, \alpha \circ \gamma\}) = \lim M$.

More explicitly, an equaliser is a morphism $\gamma \in \mathcal{K}(c,a)$ equalising morphisms $\alpha \circ \gamma = \beta \circ \gamma$ and such that given any object $o \in \operatorname{Obj}\mathcal{K}$ and any morphism $\mu : o \to a$, if m equalises α and β, then there exists a unique morphism $\nu : o \to c$ such that $\gamma \circ \nu = \mu$.

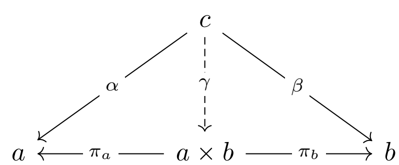


By changing the direction of equalisation, we obtain the notion of a *coequaliser*.

Definition 4.9 (Product)**.** Let $M : D \to \mathcal{K}$ be a diagram and D be discrete. A limit of M is termed a *product*, denoted $\prod_{d \in \operatorname{Obj} D} M(d)$ or $d_1 \times \cdots \times d_n$ if $\operatorname{Obj} D$ is finite.

Less generally, this means that given two objects $a, b \in \operatorname{Obj}\mathcal{K}$, a product of a and b, denoted $a \times b$, equipped with a pair of morphisms $\pi_a : a \times b \to a$ and $\pi_b : a \times b \to b$ — the only components of the natural equivalence wherewith $a \times b$ forms a limit of M — such that for every object c and every pair of morphisms $\alpha : c \to a$, $\beta : c \to b$ — a cone $(c, \{\alpha, \beta\})$ of M — there exists a unique morphism

$\gamma : c \to a \times b$ — the unique morphism guaranteed to exist between the limit $(a \times b, \{\pi_a, \pi_b\})$ and the cone $(c, \{\alpha, \beta\})$ by definition of a limit — such that the following diagram commutes:



Example 4.10.

(1) Let $\mathcal{D}$ be discrete with at least one object. In this setting, a limit is called a *product* whose apex is denoted $\prod_{d \in \mathrm{Obj}\,\mathcal{D}} M(d)$, where for each $d \in \mathrm{Obj}\,\mathcal{D}$, the limit projection $p_d : \prod_{d \in \mathrm{Obj}\,\mathcal{D}} M(d) \to M * (d)$. In $\mathcal{SET}$, $\mathcal{TOP}$, $\mathcal{GRP}$, $\mathcal{MOD}-\boldsymbol{T}$ the product is the Cartesian product on the respective settings.

Dually, we define colimits and coproducts. The apex is denoted $\coprod$.

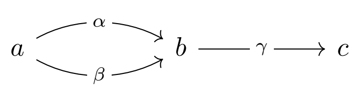
In $\mathcal{SET}$ this corresponds with disjoint unions. $(\forall d \in \mathrm{Obj}\,\mathcal{D}) M(d) \in \mathrm{Obj}\,\mathcal{SET}$. $\coprod_{d \in \ldots} M(d) = \bigcup \{M(d) \times \{d\} \mid d \in \mathrm{Obj}\,\mathcal{D}\}$.

In groups, this corresponds with free product denoted $*_{d \in \ldots} M(d)$. In $\mathcal{AB}, \mathcal{MOD} - \boldsymbol{R}$ we have the product $\bigoplus_{d \in \ldots} M(d)$.

(2) In $\mathcal{SET}$, $\mathrm{eq}(\alpha, \beta) = (\{x \in \mathrm{Dom}\,\alpha \mid \alpha(x) = \beta(x)\}, \{\gamma, \alpha \circ \gamma\})$, where γ is an identical embedding into a.

In $\mathcal{GRP}$, the apex of $\mathrm{eq}(\alpha, \beta)$ is likewise a group. In $\mathcal{AB}$,the apex of $\mathrm{eq}(\alpha, \beta)$ is $\mathrm{Ker}(\alpha - \beta)$.

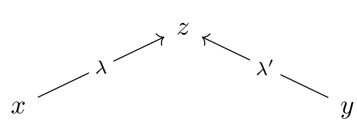
Coequalisers in $\mathcal{SET}$:



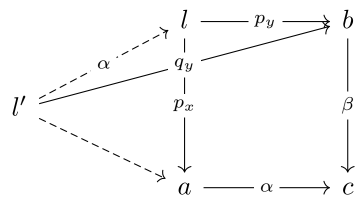
$\gamma \circ \alpha = \gamma \circ \beta$. $R \subseteq b \times b, R = \{(\alpha(x) < \beta(x)) \mid x \in a\}$. Let R^* be the least equivalence on b containing R; $c := {}^{b}\!/_{R^*}$, $\gamma(y) = [y]_{R^*}$.

Definition 4.11 (Kernel)**.** If $\mathcal{K}$ has a null object, then $\mathrm{eq}(\alpha, 0) = \mathrm{Ker}\,\alpha$, the kernel of α. Note 0 stands for the null morphism. Dually, the *cokernel* is defined by $\mathrm{coeq}(\alpha, 0) = \mathrm{Coker}\,\alpha$.

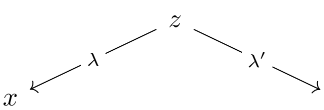
Definition 4.12 (Pullback)**.** Suppose $\mathcal{D}$ is a diagram scheme of the form:



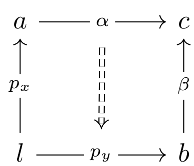
Then the limit of the diagram $M : \mathcal{D} \to \mathcal{K}$ is termed *pullback*; $(l, p) = \lim M$.



The dual term is *pushout*; it is the colimit of $M : D \to \mathcal{K}$.



Exercise 4.13. Let $(l, \{p_x, p_y\})$ be a pullback. If α is a monomorphism, then p_y is likewise monic.



Definition 4.14. Let diagram scheme $\mathcal{D}$ be thin, ordered, and upwardly closed (nahoru usporadana) and $(P, \leqslant)$ be a partially ordered set with the property $(\forall x, y \in P)(\exists z \in P) x \leqslant z \wedge y \leqslant z$.

Then the limit of $M : \mathcal{D}^{\mathrm{op}}\, \mathcal{K}$ is said to be the *inverse limit*, written $\lim_{\leftarrow}$. The colimit of M is said to be a *direct colimit*, written $\lim_{\rightarrow}$.

Example 4.15. The following two examples are, in a sense, the same.

(1) Consider the setting $(\mathbb{N}, \leqslant)$ and set $\operatorname{Obj} D = \mathbb{N}$, $a, b \in \mathbb{N}$ and $a \leqslant b \Leftrightarrow |D(a, b)| = 1$.

$$M : D \to \mathcal{AB}$$
$$a \mapsto \frac{1}{a'} \in \mathbb{Z} \leqslant \mathbb{Q}$$

Observe $\mathbb{Q}$ is the apex of the colimit of M, since

$$\mathbb{Q} = \bigcup_{a \in \mathbb{N}} \frac{1}{a!}\mathbb{Z}$$

$$Z \subseteq \frac{1}{2}\mathbb{Z} \subseteq \frac{1}{6}\mathbb{Z} \subseteq \cdots \subseteq \mathbb{Q}.$$

We have $\operatorname{colim} M = (\mathbb{Q}, i = \{i_1 \mid a \in \mathbb{N}\})$

(2) Consider the diagram

$$N : D \to \mathcal{AB}$$
$$a \mapsto \mathbb{Z}$$
$$\alpha_{ba} \mapsto \_ \cdot \frac{b!}{a!}$$

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \cdots$$

Then $\operatorname{colim} N = (\mathbb{Q}, j)$ where $j_a : \mathbb{Z} \simeq \frac{1}{a!}\mathbb{Z} = \mathbb{Q}$.

Example 4.16. Let $M : \mathcal{D}^{\mathrm{op}} \to \mathcal{TOP}$.

$$(\forall a \in \mathbb{N}) M(a) = \{z \in \mathbb{C} \mid \|z\|\} \ldots$$

Definition 4.17 (Finite Category)**.** A category $\mathcal{K}$ is said to be *finite* if both $\operatorname{Obj}\mathcal{K}$ and $\operatorname{Mor}\mathcal{K}$ are finite sets.

Definition 4.18 ((Finitely) Complete Category)**.** A category is said to be (finitely) *complete* if all (finite) limits exist — if the limits of all diagrams with any (finite) schemes exist. The dual notion is *(finite) cocompleteness.*

Theorem 4.19 (Maranda)**.** *A category $\mathcal{K}$ is (finitely) complete iff it has all (finite) products and equalisers*

Proof.

$\Rightarrow$ Trivial.

$\Leftarrow$ Let $M : \mathcal{D} \to \mathcal{K}$ be a diagram and $\mathcal{D}_o$ and $\mathcal{D}_m$ be discrete categories with $\operatorname{Obj} D_o = \operatorname{Obj}\mathcal{D}$ and $\operatorname{Obj}\mathcal{D}_m = \operatorname{Mor}\mathcal{D}$.

Define the functor Cod

$$\operatorname{Cod} : D_m \to D_0$$
$$\alpha \mapsto \operatorname{Cod}\alpha$$
$$(1_\alpha \mapsto 1_{\operatorname{Cod}\alpha})$$

Set $N : \mathcal{D}_o \to \mathcal{K}$ by

$$N(d) = M(d) \qquad \text{for each } d \in \operatorname{Obj}\mathcal{D}_o.$$

Our working category $\mathcal{K}$ admits all products whence it admits in particular

$$(t, p) \overset{\text{def}}{=} \lim N \qquad\qquad (s, q) \overset{\text{def}}{=} \lim(N \circ \operatorname{Cod})$$
$$t = \prod_{d \in \operatorname{Obj} D} M(d) \qquad\qquad s = \prod_{\lambda \in \operatorname{Mor} D} M(\operatorname{Cod}(\lambda)).$$

Lastly, define the cones

$$k_1 \stackrel{\text{def}}{=} (t, \{p_{\operatorname{Cod}\lambda} : t \to M(\operatorname{Cod}\lambda)) \mid \lambda \in \operatorname{Obj} D_m\})$$
$$k_2 \stackrel{\text{def}}{=} (t, \{M(\lambda) \circ p_{\operatorname{Dom}\lambda} : t \to M(\operatorname{Cod}\lambda) \mid \lambda \in \operatorname{Obj} \mathcal{D}_m\})$$

The latter elements in these ordered pairs really do define a natural transformation since the appropriate compatibility-conditions are met trivially — the set of morphisms on which they should be verified is empty.

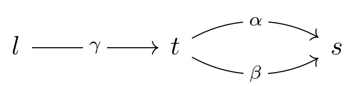
Observe

$$M(\lambda) : \operatorname{Dom} M(\lambda) = M(\operatorname{Dom}\lambda) \to M(\operatorname{Cod}\lambda) = \operatorname{Cod} M(\lambda).$$

Since $(s, q) = \lim(N \circ \operatorname{Cod})$, there exist uniquely determined morphisms $\alpha, \beta \in \mathcal{K}(t, s)$ such that

$$(\forall \lambda \in \operatorname{Mor} D)\; q_\lambda \circ \alpha = p_{\operatorname{Cod}\lambda}$$
$$q_\lambda \circ \beta = M(\lambda) \circ p_{\operatorname{Dom}\lambda}.$$

Set $\operatorname{eq}(\alpha, \beta) = (l, \{\gamma, \alpha \circ \gamma\})$.



We claim that

$$\lim M = (l, \pi) \qquad \text{where } \pi_d = p_d \circ \gamma \qquad \text{for each } \alpha \in \operatorname{Obj} D.$$

We have defined all we needed, we shall now verify our constructions to prove the claim. Denote $L = (l, \pi)$.

L is a cone. We first check (l, π) is a cone of the diagram M; for every $\lambda \in D(d, d')$ we need to check

$$(q_\lambda \circ \beta \circ \gamma = M\lambda \circ p_d \circ \gamma =) M(\lambda) \circ \pi_d \stackrel{?}{=} \pi_{d'} (= p_{d'} \circ \gamma = p_{\operatorname{Cod}\lambda} \circ \gamma = q_\lambda \circ \alpha \circ \gamma).$$

Since γ equalises α, β, we obtain $q_\lambda \circ \alpha \circ \gamma = q_\lambda \circ \beta \circ \gamma$, where $\pi'_d = \alpha$ and $\beta = M(\lambda) \circ \pi_d$. Hence L is a cone.

L is a limit. Let (l', ρ) be the cone of the diagram M. Then (l', ρ) is also the cone of the diagram N (which contains only a fraction of information stored by M). There exists, therefore, a unique morphism $\delta : l' \to t$ such that $p_d \circ \delta = \rho_d$.

We can see that δ equalises α, β since $\alpha \circ \delta = \beta \circ \delta$ holds iff for each $\lambda \in \operatorname{Mor} D$

$$(\rho_{\operatorname{Cod}\lambda} = p_{\operatorname{Cod}(\lambda)} \circ \delta =)$$
$$q_\lambda \circ \alpha \circ \delta = q_\lambda \circ \beta \circ \delta$$
$$(= M(\lambda) \circ p_{\operatorname{Dom}\lambda} \circ \delta = M(\lambda) \circ \rho_{\operatorname{Dom}\lambda})$$

by theorem ??.

But $\rho_{\mathrm{Cod}\,\lambda} = M(\lambda) \circ \rho_{\mathrm{Dom}\,\lambda}$ holds since (l', ρ) is the cone of the diagram M.

δ equalises α, β whence there exists a unique $\epsilon \in \mathcal{K}(l','))\, \gamma \circ \epsilon = \delta$. Then $\pi_d \circ \epsilon = \rho_d = p_d \circ \delta = p_d \circ \gamma \circ \epsilon$ for each $d \in \mathrm{Obj}\, D$.

ϵ is the only morphism from $\mathcal{K}(l', l)$ with this property.

QED

Remark 4.20. The proof of Theorem 4.19 may serve as a template for constructing limits in 'concrete' categories. For example, in $\mathcal{AB}$ and $\mathcal{RNG}$ with a prime p

$$\cdots \xrightarrow{f_3} \mathbb{Z}_{p^3} \xrightarrow{f_2} \mathbb{Z}_{p^2} \xrightarrow{f_1} \mathbb{Z}_p$$

where f_i is defined as taking the argument modulo p^i. The limit of such a diagram is $J_p = \left\{ g \in \prod_{i=1}^{\infty} \mathbb{Z}_{p^i} \,\middle|\, (\forall i \in \mathbb{N})\, f_i \circ g(i+1) = g(i) \right\}$, the set of all p-adic numbers.

The product $\prod_{i=1} \mathbb{Z}_{p^i}$ would be our t from the proof of Theorem 4.19.

Remark 4.21. The dual version for colimits, coproducts, and coequalisers in Theorem 4.19 holds.

Theorem 4.22 (Mitchell)**.** *The category $\mathcal{K}$ has all limits iff it has all products and pullbacks.*

Proof. ($\Rightarrow$) Trivial since a pullback is a special case of a limit.

($\Leftarrow$) We shall use Theorem 4.19. Suppose we have

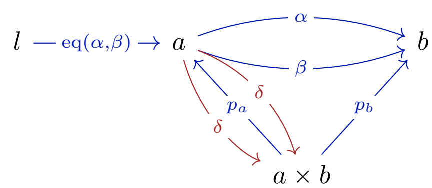
$$l \xrightarrow{\mathrm{eq}(\alpha,\beta)} a \xrightarrow{\beta} b$$

We are trying to find $\mathrm{eq}(\alpha, \beta)$. Consider the product $(a \times b, p)$ where $p = \{p_a, p_b\}$, where $p_a : a \times b \to a$ and $p_b : a \times b \to b$. This is a little bit formally unwieldy, since if $a = b$, then these two morphisms are named the same even if they are different.

Consider the morphisms $\gamma, \delta : a \to a \times b$, where

$$p_a \circ p = p_a \circ \delta = 1_a \qquad\qquad p_b \circ \gamma = \delta, \quad p_b \circ \delta = \beta$$

Observe these equalities uniquely determine them since we have defined them on all cone-projections as per Theorem 4.5. Moreover, $(a, \{1_a, \alpha\}), (a, \{1_a, \beta\})$ are cones of the product-diagram.



Consider the following pullback:

$$\begin{array}{ccc} l & \xrightarrow{q_\delta} & a \\ \big\downarrow{\scriptstyle q_\gamma} & & \big\downarrow{\scriptstyle \gamma} \\ a & \xrightarrow{\delta} & a \times b \end{array}$$

Then $q_\gamma = p_a \circ (\delta \circ q_p) = p_\alpha \circ \gamma q_\delta = q_\delta$. Let $q := q_\delta$. We claim $q : l \to a$ is an equaliser of the morphisms α, β.

(1) $\alpha \circ q = p_b \circ \gamma \circ q = p_b \circ \delta \circ q = \beta \circ q$.

(2) $q' : l' \to a$ equalises the morphisms α, β. We will show $\gamma \circ q' = \delta \circ q'$ on the limit (product) projections whence they will be equal by theorem ???.

$$\underbrace{p_a \circ \gamma}_{1_a} \circ q' = q' = \underbrace{p_a \circ \delta}_{1_a} \circ q' p_b \circ \gamma \circ q' = \alpha \circ q' = \beta \circ q' = p_b \circ \delta \circ q'$$

Since l' is the apex of the pullback-cone, and thus by definition of a pullback, there exists a unique morhpism $\epsilon : l' \to l$ such that $q \circ \epsilon = q'$. We have verified $q = \mathrm{eq}(\alpha, \beta)$.

QED

Exercise 4.23. Show that a category has finite limits iff it has a terminal objects and all pullbacks.

4.1 Complete Small Categories

Example 4.24. Let $\mathcal{K}$ be a thin, small category. Is $\mathcal{K}$ complete? $\mathcal{K}$ has all equalisers since all are of the form

$$a \xrightarrow{1_a} a \overset{\alpha}{\underset{\alpha}{\rightrightarrows}} b$$

and $\mathrm{eq}(\alpha, \alpha) = 1_a$. It also has all products in $\mathcal{K}$: Consider the cone $M : D \to \mathcal{K}$, where D is discrete. In the special case $D = \varnothing$: we have the terminal object t, which is the greatest (up to isomorphism) in $\mathrm{Obj}\,\mathcal{K}$ as quasi-ordered set by the relation ‘an arrow leads to me from X’.

More generally, $\prod_{d \in \mathrm{Obj}\, D} M(d) = \inf \{M(d) \,|\, d \in \mathrm{Obj}\, D\}$ up to isomorphism, if it exists.

It follows by 4.19, $\mathcal{K}$ is complete iff $\mathcal{K}$ has all infima including $\inf \varnothing =:$ the greatest element up to isomorphism.

Let $S \subseteq \mathrm{Obj}\, \mathcal{K}$ be a skeleton. S determines a complete subcategory $\mathcal{S}$, termed a *skeletal subcategory*, where $\mathrm{Obj}\, \mathcal{S} = S$. Note $\mathcal{S}, S$ are ordered sets. Although, $\mathcal{S} \subseteq \mathcal{K}$, $\mathcal{S}, \mathcal{K}$ are equivalent categories. To see this, note $\mathrm{Id} : \mathcal{S} \to \mathcal{K}$ is an identity-embedding as it is injective, faithful, full, and essentially surjective.

It follows then $\mathcal{K}$ is complete iff $\mathcal{S}$ is complete. An ordered set with all infima is termed a *complete lattice*.

Observe complete lattices have all suprema. Conversely, the ordered set with all suprema is a complete lattice.

Proof. We shall prove only the first claim. $X \subseteq L$ and

$$\sup X = \inf \{y \in L \,|\, (\forall x \in X)\, x \leqslant y\}\,.$$

Note every y is an upper bound of X. QED

As a corollary, we know a thin category is complete iff it is cocomplete.

Exercise 4.25. Let $\mathcal{K}, \mathcal{H}$ be thin, small categories, which 'are' ordered sets. Then $\mathcal{K} \simeq \mathcal{H}$ iff these ordered sets are isomorphic.

Note both $\mathcal{K}$ and $\mathcal{H}$ are their own skeletons; they are *skeletal* categories.

Theorem 4.26. *Let $\mathcal{K}$ be a small category. Then the following are equivalent:*

(1) $\mathcal{K}$ is complete/cocomplete.

(2) $\mathcal{K}$ has all products/coproducts.

(3) $\mathcal{K}$ is thin and equivalent with a complete lattice.

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Trivial. $(\mathbf{3}) \Rightarrow (\mathbf{1})$ Example 4.24 $(\mathbf{2}) \Rightarrow (\mathbf{3})$. We only need to show $\mathcal{K}$ is thin in view of ???. Towards a contradiction, suppose there are $\alpha, \beta : a \to b$ in $\mathcal{K}$ where $\alpha \neq \beta$. Denote D a discrete category, where $\mathrm{Obj}\, D = \mathrm{Mor}\, \mathcal{K}$. Let $M : D \to \mathcal{K}$ be defined by $(\forall d \in \mathrm{Obj}\, D)\, M(d) := b$; i.e. $M = \Delta_b$. Denote $\lim M = (s, p)$, where $s \in \mathrm{Obj}\, \mathcal{K}$.

For each $Y \subseteq \mathrm{Mor}\, \mathcal{K}$ let $\gamma_Y : a \to s$ be defined on its projections by

$$p_d \circ \gamma_Y = \begin{cases} \alpha & \text{if } \alpha \in Y \\ \beta & \text{otherwise} \end{cases}$$

For $Y, Z \subseteq \mathrm{Mor}\, \mathcal{K}$ such that $Y \neq Z$, we have $\gamma_Y \neq \gamma_Z$. Let, for example, $d \in Y \backslash Z$, then $p_d \circ p_Y = \alpha \neq \beta = p_d \circ \gamma_Z$.

The set $\operatorname{Mor}\mathcal{K} \supseteq \mathcal{K}(a,s) \supseteq \{\gamma_Y \,|\, Y \subseteq \operatorname{Mor}\mathcal{K}\}$ is a set comprising more morphisms than are in $\operatorname{Mor}\mathcal{K}$. We are using Cantor's theorem which states the powerset of a set has a greater cardinality than the original set. The argument we have employed has in fact a hidden diagonal argument. QED

Definition 4.27 (Bicomplete Category)**.** A category $\mathcal{K}$ is said to be *bicomplete* if it complete and cocomplete.

Example 4.28.

(1) *Bicomplete Categories.* $\mathcal{SET}, \mathcal{AB}, \mathcal{MOD} - \boldsymbol{R}, \mathcal{GRP}, \mathcal{DAB}$. Note that $\mathcal{DAB}$ has different equalisers (and kernels) to $\mathcal{AB}$.

(2) Let $\mathcal{K}$ be the thin category of all ordinal numbers. $\mathcal{K}$ is cocomplete, but not complete. The coproduct of two ordinal products is their supremum. The reason why the category is not complete is that there is no terminal object; i.e. there is no largest ordinal number.

(3) Let $\boldsymbol{R}$ be a non-Noetherian commutative ring (e.g. $\boldsymbol{R} = \mathbb{Q}[x_1, x_2, \dots])$, then $\mathcal{K}$ has all products, but it does not have kernels (equalisers).

5 Limit & Colimit Invariance

Lemma 5.1. *Let $M : D \to \mathcal{K}$ be a diagram with a cone (a,p) and a functor $F : \mathcal{K} \to \mathcal{H}$, then $(F(a), Fp = \{F(p_d) \,|\, d \in \operatorname{Obj} D\})$ is a cone of $D \to \mathcal{H}$.*

Proof. Note $F_p : F \circ \Delta_a \to F \circ M$ and $F \circ \Delta_a = \Delta_{F(a)} : D \to \mathcal{H}$. QED

Definition 5.2 (Limit Preservation)**.** Let $M : D \to \mathcal{K}$ be a diagram with $\lim M = (a,\pi)$ and $F : \mathcal{K} \to \mathcal{H}$ be a functor. It is said F *preserves* the limit (a,π) if $(F(a), F\pi) = \lim(F \circ M)$.

Moreover, F is said to preserve limits, products, or equalisers if it preserves all limits, products, or equalisers respectively.

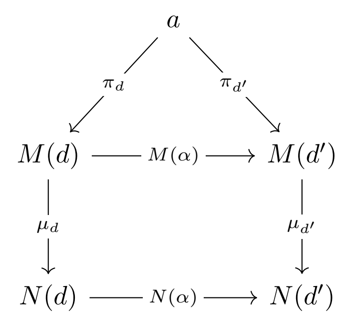
Theorem 5.3. *Let $M : D \to \mathcal{K}$ be a diagram.*

(1) If $N : D \to \mathcal{K}$ is another diagram naturally equivalent with M by $\mu : M \to N$, then given any limit (a,π) of M, the cone $(a, \mu \circ \pi)$ is a limit of N.

(2) If $F, G : \mathcal{K} \to \mathcal{H}$ are naturally equivalent functors and $(a,\pi) = \lim M$, then F preserves (a,π) iff G preserves (a,π).

Proof.

(1) The pair $(a, \mu \circ \pi)$ is clearly a cone of N as evidenced by the following diagram.



We will show it is a limit. By the same token, if (b, ρ) is any cone of N, then $(b, \mu^{-1} \circ \rho)$ is a cone of M. Since $(a, \pi) = \lim M$, there is a unique $\alpha \in \mathcal{K}(b, a)$ such that

$$(\forall d \in \operatorname{Obj} D)\, \pi_d \circ \alpha = \mu_d^{-1} \circ \rho_d \Rightarrow (\forall d \in \operatorname{Obj} D)\, \overbrace{(\mu_d \circ \pi_d)}^{(\mu\circ\pi)_d} \circ \alpha = \rho_d$$
$$\Rightarrow (a, \mu \circ \pi) = \lim N.$$

(2) The claim's symmetry allows for only one implication being shown to prove equivalence. Let $\iota : F \to G$ be a natural equivalence and F preserve (a, π); i.e. $(F(a), F\pi) = \lim(F \circ M)$. The fact $\iota_M : F \circ M \to G \circ M$ is a natural equivalence follows from the diagram below.

Putting $\mu := \iota_M$ in (1) yields

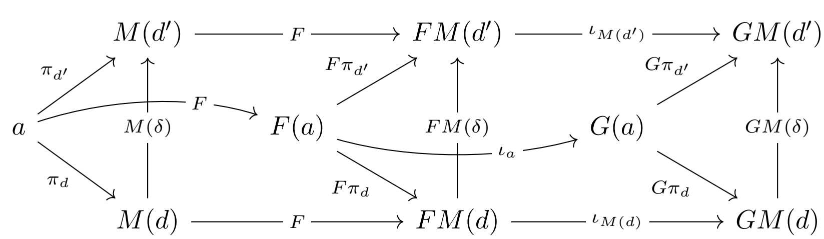
$$(F(a), \iota_M \circ F\pi) = \lim(G \circ M).$$

Writing out explicitly the components of the natural transformation above gives

$$\iota_M \circ F\pi = \left\{ \iota_{M(d)} \circ F(\pi_d) \,\middle|\, d \in \operatorname{Obj} D \right\},$$

where $\iota_{M(d)} \circ F(\pi_d = G\pi_d \circ \iota_a$ (as seen in the diagram), whence (recalling ι_a is an isomorphism and hence invertible)

$$\begin{aligned} \lim(G \circ M) &= (F(a), \iota_M \circ F\pi) \\ &= (F(a), \{G\pi_d \circ \iota_a \mid d \in \operatorname{Obj} D\}) \\ &= (G(a), G\pi). \end{aligned}$$



QED

Theorem 5.4. *Let $\mathcal{K}$ be a category. For any $c \in \operatorname{Obj}\mathcal{K}$ we denote $F_c : \mathcal{K} \to \mathcal{SET}$ functor $\mathcal{K}(c, -)$. Let (a, π) be a cone of the diagram $M : D \to \mathcal{K}$. Then $(a, \pi) = \lim M$ iff $(\forall c \in \operatorname{Obj}\mathcal{K})\,(F_c(a), F_c\pi) = \lim(F_c \circ M)$.*

Proof. ($\Leftarrow$) We shall show from definition that $(a, \pi) = \lim M$. Let (b, ρ) be any cone of M. By assumption, $(F_b(a), F_b\pi) = \lim(F_b \circ M)$. At the same time $(F_b(b), F_b\rho)$ is a cone of $F_b{\circ}M$. There exists then a unique $g \in \mathcal{SET}(F_b(b), F_b(a))$ such that $F_b(\pi_d) \circ g = F_b(\rho_d)$ for every $d \in \operatorname{Obj} D$). Recall $F_b(b) = \mathcal{K}(b, b) \ni 1_b$.

Set $\alpha := g(1_b) \in \mathcal{K}(b, a)$. We will show α is the only such that $\pi_d \circ \alpha = \rho_D$ for every $d \in \operatorname{Obj} D$.

(1) For $d \in \operatorname{Obj} D$ we have $\pi_d \circ \alpha = \pi_D \circ g(1_b) = \mathcal{K}(b, \pi_d)(g(1_b)) = (F_b(\pi_d) \circ g)(1_b) = F_b(\rho_d)(1_b) = \rho_d \circ 1_B = \rho_d$.

(2) *Uniqueness.* If $\alpha' \neq \alpha$ were such that $(\forall d \in \operatorname{Obj} D)\, \pi_d \circ \alpha' = \rho_d$, then $g' \in \mathcal{SET}(F_b(b), F_b(a))$ such that $G'(1_b) = \alpha'$, $g'(x) = g(x)$ for $x \neq 1_b$, satisfies $F_b(\pi_d) \circ g' = F_b(\rho_d)$. A contradiction with the uniqueness of g.

($\Rightarrow$) Suppose $(a, \pi) = \lim M$. We will show that for each $c \in \operatorname{Obj}\mathcal{K}$, F_c preserves (a, π) such that $(F_c(a), F_c\pi) = \lim(F_c \circ M)$. Let (s, σ) be a cone of the diagram $F_c \circ M$.

We define $\alpha \in \mathcal{SET}(s, F_C(a))$ elementwise. For any $x \in s$ and any $d \in \operatorname{Obj} D$ we consider $\sigma_d : s \to F_c(M(d))\,\mathcal{K}(c, M(d))$, more precisely $\sigma_d(x) \in \mathcal{K}(c, M(d))$.

We claim $(c, \{\sigma_d(x) \mid d \in \operatorname{Obj} D\})$is a cone of the diagram M. We will verify the compatibility conditions hold.

$\forall\lambda \in D(d, d')\, M(\lambda) \circ \sigma_d(x) ?{=} \sigma_{d'}(x)$.

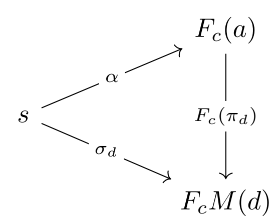
Since σ is a natural transformation,we know that $\forall\lambda \in D(d, d')(F_c{\circ}M)(\lambda){\circ}\sigma_d = \sigma_{d'}$. But $(F_C{\circ}M) = F_c(M(\lambda)){\circ}\sigma_d \overset{x\in s}{\Rightarrow} F_c((\lambda))(\sigma_d(x)) = m(\lambda){\circ}\sigma_d(x) = \sigma_{d'}(x)$.

Thus indeed $M(\lambda) \circ \sigma_d(x) = \sigma_{d'}(x)$.

Since $(a, \pi) = \lim M$, there exists a unique $\alpha(x) \in \mathcal{K}(c, a)$ such that $\pi_d{\circ}(\alpha(x)) = \sigma_d(x)$ for every $d \in \operatorname{Obj} D$. Since $x \in s$ was arbitrary, we have thus defined some $\alpha \in \mathcal{SET}(s, F_c(a))$. For any $d \in \operatorname{Obj} D$, then

$$(\forall x \in s)(F_c(\pi_d) \circ \alpha)(x) = F_c(\pi_d)(\alpha(x)) = \pi_d \circ (\alpha(x)) = \sigma_d(x). \tag{1}$$

It remains to show $\alpha \in \mathcal{SET}(x, F_c(a))$ is the only one satisfying Equation (1).



If $\alpha \in \mathcal{SET}(s, F_C(a))$ were different to α, then there would exist some $x \in s$ such that $\alpha'(x) \neq \alpha(x)$, contradicting the definition of $\alpha(x)$.

Observe that we have a different cone for different x. QED

Corollary 5.4.1.

(1) Representable functors preserve limits (as follows from Theorem 5.4 and Theorem 5.3)

(2) A functor $U : \mathcal{H} \to \mathcal{K}$ preserves limits iff $(\forall c \in \operatorname{Obj} \mathcal{K})\ \mathcal{K}(c, -) \circ U : \mathcal{H} \to \mathcal{SET}$ preserves limits.

6 Adjoint Functors

We remark the term *adjoint functor* first appeared in an article 'Adjoint Functors' by Dr Kan in TAMS.

Definition 6.1 (Adjoint Functor, Adjunction)**.** Let $F : \mathcal{K} \to \mathcal{H}$ and $U : \mathcal{H} \to \mathcal{K}$ be covariant functors. F is termed a *left adjoint* of U (and conversely U is termed *a right adjoint* of F) if the functors $\mathcal{H}(F(-), -), \mathcal{K}(-, U(-)) : \mathcal{K}^{\text{op}} \times \mathcal{H} \to \mathcal{SET}$ are naturally equivalent.

Note $\mathcal{H}(F, -)$ is to be interpreted as $\mathcal{H}(-, -) \circ (F \times \operatorname{Id}_{\mathcal{H}})$ but the intuitive view is preferable.

By the *adjunction* of F, U we understand the ordered triple (F, U, ϕ) where $\phi : \mathcal{H}(F(-), -) \rightharpoonup \mathcal{K}(-, U(-))$ is a natural equivalence. The fact is denoted $F \dashv_\phi U$ or more briefly $F \dashv U$.

We shall illustrate this for a natural equivalence $\phi = \{\phi_{a,b} \mid a, b \in \operatorname{Obj}(\mathcal{K}^{\text{op}} \times \mathcal{H})\}$. Note that since the target-category is $\mathcal{SET}$, $\phi_{a,b}$ being ismorphisms implies their being mere bijections (in $\mathcal{SET}$).

$$\begin{array}{ccccc} a & a & \mathcal{H}(F(a), b) & \xrightarrow{\mathcal{H}(F(\alpha),\beta)} & \mathcal{H}(F(a'), b') \\ \alpha\Big\uparrow & \Big\uparrow\beta & \Big\downarrow \phi_{a,b} & & \Big\downarrow \phi_{a',b'} \\ a' & b' & & & \\ \mathcal{K} & \mathcal{H} & \mathcal{K}(a, U(b)) & \xrightarrow{\mathcal{K}(\alpha,U(\beta))} & \mathcal{K}(a', U(b')) \end{array}$$

Example 6.2. Let $\mathcal{K} = \mathcal{H} = \mathcal{MOD}_{\boldsymbol{R}}$, where $\boldsymbol{R}$ is a commutative ring. Fix $B \in \operatorname{Obj} \mathcal{K}$. Then $F \dashv U$, where $F = B \otimes_{\boldsymbol{R}} -, U = \mathcal{H}(B, -)$.

Notation 6.3. Let $F : \mathcal{K} \to \mathcal{H}$ be a functor. Then $F^{\text{op}} : \mathcal{K}^{\text{op}} \to \mathcal{H}^{\text{op}}$ is a covariant functor where

$$(\forall a \in \operatorname{Obj} \mathcal{K})\ F^{\text{op}}(a) = f(a) \qquad (\forall \alpha \in \operatorname{Mor} \mathcal{K})\ F^{\text{op}}(\alpha) = F(\alpha).$$

Indeed, F^{op} is indistinguishable from F by its assignment of value. We, nevertheless, need to treat them as separate formal entities.

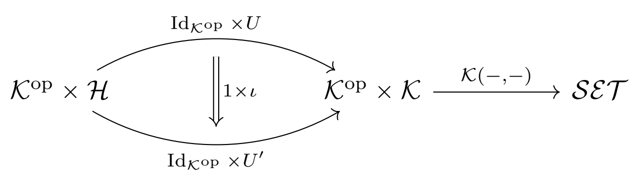
Exercise 6.4. $F \dashv_\phi U \Leftrightarrow U^{\mathrm{op}} \dashv_{\phi^{-1}} F^{\mathrm{op}}$. Note ϕ^{-1} exists because ϕ is a natural equivalence (and hence is composed of isomorphisms).

Theorem 6.5. *Let $F : \mathcal{K} \to \mathcal{H}$ and $U : \mathcal{H} \to \mathcal{K}$ be covariant functors and F', U' be naturally equivalent with F, U respectively. Then*

$$F \dashv U \Leftrightarrow F' \dashv U'.$$

Proof. Due to Exercise 6.4, it suffices to prove the claim for $F = F'$. Were it not so, we could split the proof into two steps: first showing the claim for $F = F'$ and the second for $U = U'$. But in this two-step process, the first demonstration is sufficient since we could invoke the exercise and be done.

Fix some natural equivalence $\iota : U \to U'$. Note $1 \times \iota$ is still a natural equivalence and consequently we arrive at the following diagram.



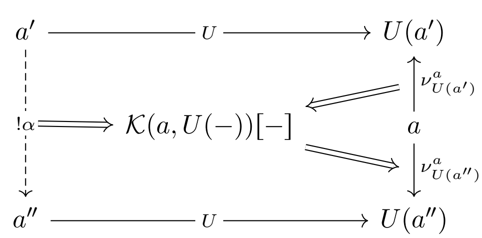
Since functors preserve isomorphisms, we obtain the natural equivalence

$$\begin{aligned} \mathcal{K}(-,-)(1 \times \iota) = \mathcal{K}(-,\iota) &: \mathcal{K}(-, U(-)) \to \mathcal{K}(-, U'(-)) \\ &\overset{\mathrm{def}}{\Leftrightarrow} F \dashv U'. \end{aligned}$$

QED

Definition 6.6 (Free Object)**.** Let $U : \mathcal{H} \to \mathcal{K}$ and $a \in \mathrm{Obj}\,\mathcal{K}$. The pair $(a', \eta^a_{U(a')}) \in \mathrm{Obj}\,\mathcal{H} \times \mathrm{Obj}\,\mathcal{K}$ is termed a *free object* over a with respect to U if it is the universal element (pair) of the functor $\mathcal{K}(a, U(-)) = \mathcal{K}(a, -) \circ U$.

In greater detail, given any pair $(a'', \eta^a_{U(a'')}) \in \mathrm{Obj}\,\mathcal{H} \times \mathrm{Obj}\,\mathcal{K}$ there exists a unique $\alpha \in \mathcal{H}(a', a'')$ such that $\mathcal{K}(a, U(\alpha))[\eta^a_{U(a')}] = \eta^a_{U(a'')}$.



Theorem 6.7 (Characterisation of Adjunction)**.** *Let $U : \mathcal{H} \to \mathcal{K}$ be a functor. Then the following are equivalent.*

(1) U is the right adjoint (or admits a left adjoint) of $F : \mathcal{K} \to \mathcal{H}$; $F \dashv U$.

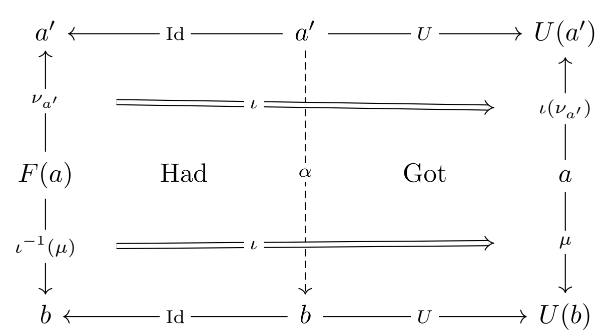
(2) For each $a \in \operatorname{Obj}\mathcal{K}$, the functor $\mathcal{K}(a, U(-))$ is representable (i.e. admits a universal pair).

(3) There exists $F : \mathcal{K} \to \mathcal{H}$ and $\eta \in \operatorname{Nat}(1_{\mathcal{K}}, U \circ F)$ such that $(F(a), \eta_a)$ is a free object. (Then $F \dashv U$ and η is termed the unit of adjunction).

(3') There exists $F : \mathcal{K} \to \mathcal{H}$ and $\epsilon \in \operatorname{Nat}(F \circ U, 1_{\mathcal{H}})$ such that $(U(b), \epsilon_b)$ is a universal element of a contravariant functor $\mathcal{H}(F(-), b)$, for each $b \in \operatorname{Obj}\mathcal{H}$ (then $F \dashv U$ and ϵ is termed the counit of adjuction).

Proof. (**1**) $\Rightarrow$ (**2**) Let $\phi : \mathcal{H}(F(-), -) \to \mathcal{K}(-, U(-)$ be a natural equivalence (which exists by assumption) and fix some $a \in \operatorname{Obj}\mathcal{K}$. Then after partial substitution $\mathcal{K}(F(a), -), \mathcal{K}(a, U(-))$ are still naturally equivalent functors from $\mathcal{H}$ to $\mathcal{SET}$ as one merely fixes one argument, namely a, in the binary components of ϕ; denote this natural equivalence ι. We know $\mathcal{K}(F(a), -) = \mathcal{K}(a', -)$ is representable functor, whence $\mathcal{K}(a, U(-))$ is likewise representable.

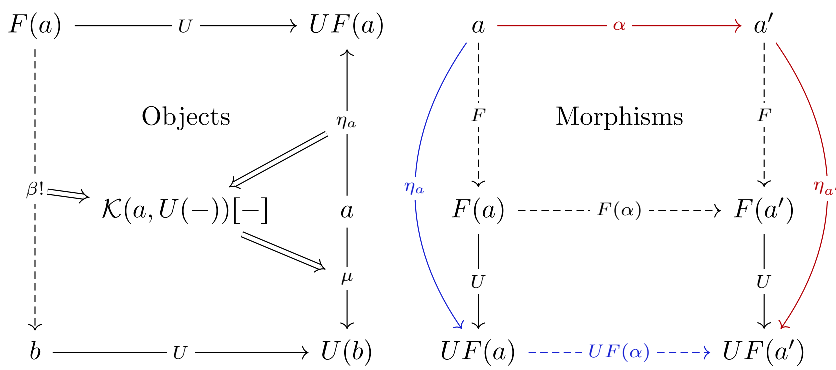
In fact, if $(a', \nu_{a'})$ is a universal element of $\mathcal{H}(F(a), -)$, the pair $(a', \iota^{-1} \circ \nu_{a'})$ is a universal element of $\mathcal{K}(a, U(-))$. To see this, let $b \in \operatorname{Obj}\mathcal{H}$ and $\mu \in \mathcal{K}(a, U(b)$. The existence of the unique morphism $\alpha : a' \to b$ is evident from the following diagram (note it commutes due to the universal property of ι):



(**2**) $\Rightarrow$ (**3**) We define F piecewise on objects and on morphisms.

Objects. For every $a \in \operatorname{Obj}\mathcal{K}$ let $(F(a), \eta_a)$ be a free object over a with respect to U; we know it exists since we assume the existence of a universal pair of $\mathcal{K}(a, U(-))$, which is by definition is a free object over a with respect to U.

Morphisms. For every $\alpha : a \to a'$ we define $F(\alpha)$ as the only morphism from $\mathcal{H}(F(a), F(a'))$ such that $\eta_{a'} \circ \alpha = UF(\alpha) \circ \eta_a$ from the definition of a universal pair. Thus, F is defined on morphisms as well.



We shall verify F is a functor.

Unity. We want to show $F(1_a) = 1_{F(a)}$. By definition of F, $F(1_a) \in \mathcal{H}(F(a), F(a))$ is the unique morphism with

$$(\eta_a =)\eta_a \circ 1_a = U \circ F(1_a) \circ \eta_a.$$

Clearly $\eta_a = 1_{U\circ F(a)} \circ \eta_a = U(1_{F(a)}) \circ \eta_a$ by virtue of U being a functor. The equality $U(1_{F(a)}) \circ \eta_a = U \circ F(1_a) \circ \eta_a$ necessitates $F(1_a) = 1_{F(a)}$ by uniqueness of $F(1_a)$.

Composition. We need to check $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$ for an arbitrary $\beta : a' \to a''$. It follows from definition $F(\beta \circ \alpha)$ is the unique morphism with

$$\nu_{a''} \circ (\beta \circ \alpha) = U \circ F(\beta \circ \alpha)\eta_a.$$

Simultaneously

$$\begin{aligned} \eta_{a''} \circ \beta \circ \alpha &= U(F(\beta) \circ F(\alpha)) \circ \eta_a \\ &= \underbrace{UF(\beta) \circ \underbrace{UF(\alpha) \circ \eta_a}_{\eta_{a'}\circ\alpha}}_{\eta_{\tilde{a}}\circ\beta\circ\alpha}. \end{aligned}$$

Thus the uniqueness of F forces the desired equality.

In view of F being a functor, it is clear from the morphisms-diagram above $\eta = \{\eta_a \mid a \in \operatorname{Obj}\mathcal{K}\}$ is a natural transformation (for the coloured subdiagram) of $1_{\mathcal{K}}$ and $U \circ F$. The claim then follows by construction and, therefore, $(F(a), \nu_a)$ is a free object for every $a \in \operatorname{Obj}\mathcal{K}$.

(**3**) $\Rightarrow$ (**1**) We begin with the functors $U : \mathcal{H} \to \mathcal{K}$, $F : \mathcal{K} \to \mathcal{H}$ and the unit of adjunction η.

We introduce the system of morphisms $\phi : \mathcal{H}(F(-), -) \to \mathcal{K}(-, U(-))$ and define its components by setting for each $(a, b) \in \mathcal{K}^{\mathrm{op}} \times \mathcal{K}$ and each $\gamma \in \mathcal{H}(F(a), b)$,

$$\mathcal{K}(a, U(b)) \ni \phi_{a,b}(\gamma) = U(\gamma) \circ \eta_a.$$

The claim is ϕ is a natural transformation. To show this, we need to verify its components are isomorphisms (or rather bijections) and the compatibility conditions.

We begin with the latter.

$$\begin{array}{ccc}
\mathcal{H}(F(a), b) & \xrightarrow{\phi_{a,b}} & \mathcal{K}(a, U(b)) \\
\Big\downarrow{\scriptstyle \mathcal{H}(F(\alpha),\beta)} & & \Big\downarrow{\scriptstyle \mathcal{K}(\alpha,U(\beta))} \\
\mathcal{H}(F(a'), b') & \xrightarrow{\phi_{a',b'}} & \mathcal{K}(a', U(b'))
\end{array}$$

$\circlearrowright$ We have

$$\begin{aligned}
\mathcal{K}(\alpha, U(\beta)) \circ \phi_{a,b}(\gamma) &= cat(\alpha, F(\beta))(U(\gamma) \circ \eta_a) \\
&= U(\beta) \circ U(\gamma) \circ \eta_a \circ \alpha.
\end{aligned}$$

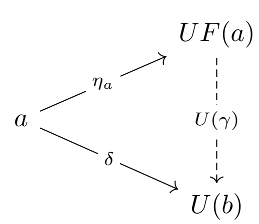
$\circlearrowleft$ Similarly,

$$\begin{aligned}
\phi_{a',b'}(\beta \circ \gamma \circ F(\alpha)) &= U(\beta \circ \gamma \circ F(\alpha)) \circ \eta_{a'} \\
&= U(\beta) \circ U(\gamma) \circ UF(\alpha) \circ \eta_{a'} \\
&= U(\beta) \circ U(\gamma) \circ \eta_a \circ \alpha,
\end{aligned}$$

where the last equality follows from the coloured diagram.

It remains to show $\phi_{a,b}$ is a bijection (an isomorphism in $\mathcal{SET}$); i.e. that for each $\delta \in \mathcal{K}(a, U(b))$ there exists a *unique* preimage γ such that $\delta = U(\gamma) \circ \eta_a$.

This is guaranteed by virtue of (a, η_a) being free. To see this, take the pair (b, δ): there exists a unique $\gamma \in \mathcal{H}(F(a), b)$ such that $U(\gamma) \circ \eta_a = \delta$.



(1) → **(3′)** Follows from the duality principle; i.e. if we consider instead of $\mathcal{K}, \mathcal{H}, U$ their opposites $\mathcal{K}^{\text{op}}, \mathcal{H}^{\text{op}}, U^{\text{op}}$.

When dualising, the unit is replaced by the counit and vice versa. QED

Corollary 6.7.1.

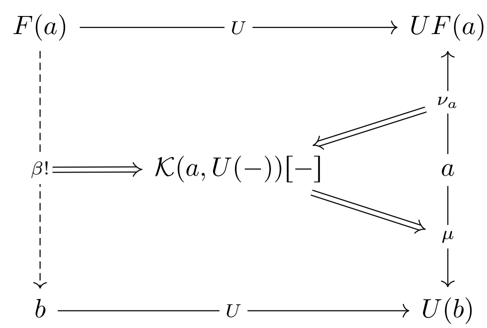
(1) *The right adjoint preserves limits. The left adjoint preserves colimits.*

(2) *Let $F : \mathcal{K} \to \mathcal{H}$, $U : \mathcal{H} \to \mathcal{K}$ witnesses $\mathcal{H}, \mathcal{K}$ are equivalent categories. Then $F \dashv U, U \dashv F$.*

Proof.

(1) Immediately by (2) of Theorem 6.7 coupled with Theorem 5.3 and Theorem 5.4. In short, we know representable functors preserve limits.

(2) Let $\eta : 1_{\mathcal{K}} \to UF$ be a natural equivalence (recall F, U being witnesses for $\mathcal{H} \simeq \mathcal{K}$ simply means $FU = 1_{\mathcal{K}}$ and $UF = 1_{\mathcal{H}}$. Theorem 3.28 implies F, U are full, faithful and essentially surjective. We wish to show η is a unit of adjunction of $F \dashv U$; i.e. that for each $a \in \operatorname{Obj} \mathcal{K}$ the pair $(F(a), \eta_a)$ is a free object. To this end, let (b, μ) be arbitrary such that $\mu \in \mathcal{K}(a, U(b))$.



The morphism η_a is invertible and therefore $\mu \circ \eta_a^{-1} \in \mathcal{K}(UF(a), U(b))$. The fullness of U implies $\mu \circ \eta_a^{-1} = U(\beta)$ for some $\beta \in \mathcal{H}(F(a), b)$ The faithfulness of U implies such a β is unique.

This proves $F \dashv U$. The converse implication that $U \dashv F$ follows from the claim's symmetry.

QED

Exercise 6.8. Using the units of adjunction, show

$$(F \dashv U \ \wedge \ F \dashv G) \to U \simeq G.$$

Lemma 6.9. *The identity-functor $1_{\mathcal{SET}} : \mathcal{SET} \to \mathcal{SET}$. Its universal pair is $(\{*\}, *)$. Consequently, $1_{\mathcal{SET}} \simeq \mathcal{SET}(\{*\}, -)$.*

Theorem 6.10. *Let $U : \mathcal{H} \to \mathcal{SET}$. Then U is the right adjoint iff U is representable and $\mathcal{H}$ admits coproducts $\coprod_{x \in a} c$, where $c \in \operatorname{Obj} \mathcal{H}$ is (some) such that $U \simeq \mathcal{H}(c, -)$; for any set a (permissibly empty).*

Proof. $\Rightarrow$ Let $F : \mathcal{SET} \to \mathcal{H}$ be such that $F \dashv U$. In particular, $s = \{*\} \in \operatorname{Obj} \mathcal{SET}$, whence by Lemma 6.9

$$\mathcal{H}(F(s), -) \simeq \mathcal{SET}(s, U-) = \mathcal{SET}(s, -) \circ U \simeq 1_{\mathcal{SET}} \circ U = U.$$

Set $c = F(s)$ and observe $U \simeq \mathcal{H}(c, -)$; i.e. U is representable. F preserves colimits, hence it also preserves coproducts of the form $\coprod_{x \in a} s$, whence $\mathcal{H}$ admits coproducts $\coprod_{x \in a} c$.

$\Leftarrow$ is left as an exercise. QED

6.1 Special Cases of Adjunction

Definition 6.11 (Free Functor)**.** Let $U : \mathcal{H} \to \mathcal{K}$ be a forgetful functor (e.g. $\mathcal{H} = \mathcal{GRP}$ and $\mathcal{K} = \mathcal{SET}$ or $\mathcal{H} = \mathcal{TOPGRP}$ and $\mathcal{K} = \mathcal{GRP}$) Then the left adjoint of U, if it exists, is termed a *free functor* of U.

Example 6.12. Let $\mathcal{K} = \mathcal{SET}$ and $\mathcal{H} = \mathcal{SMG}$. Then for $X \in \operatorname{Obj} \mathcal{SET}$, $F(X)$ is a so-called *free group* over X. Its elements are words over the alphabet X and its operations are concatenations of words.

Similarly for $\mathcal{H}$. Instead of words one considers 'reduced words' which are words over the alphabet $X \cup X^{-1}$.

Definition 6.13 (Reflective & Coreflective Categories)**.** Let $U : \mathcal{H} \to \mathcal{K}$ be an embedding-functor of the subcategory $\mathcal{H}$ into $\mathcal{K}$. If U admits a left adjoint F, then F is termed a *reflector* of U.

Free objects $(F(a), \eta_a)$ of $a \in \operatorname{Obj} \mathcal{K}\, in \mathcal{H}$ are then termed its *reflexions.*

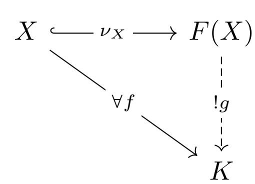
Example 6.14.

(1) Let $\mathcal{K} = \mathcal{DAB}$ and $\mathcal{H} = \mathcal{AB}$, $\epsilon_A : D \hookrightarrow A$, where D is the largest divisible group of A. Then $D := \sum_{B \leqslant A} B$.

(2) Let $\mathcal{K} = \mathcal{GRP}, \mathcal{H} = \mathcal{SYMGRAPH}$ a full subcategory of symmetric graphs. For the graph (X, R) its reflexions are of the form $((X, R \cup R^{-1}), \eta_{(X,R)})$, where $\eta_{(X,R)}$ is the canonical embedding of (X, R) into $(X, R \cup R^{-1})$.

(3) Let $\mathcal{K} = \mathcal{TICH}$ be the category of Tychonoff topological spaces and $\mathcal{H} = \mathcal{HCOMP}$ be the category of complete Hausdorff spaces.

Let $X \in \operatorname{Obj} \mathcal{TICH}$ and $F(X) = \beta X$ be its 'beta-cover'.



(4) Let $\mathcal{K} = \mathcal{UNIF}$ be the category of uniform topological spaces with uniformly continuous maps and $\mathcal{H} = \mathcal{CUNIF}$ be the category of complete uniform spaces

Definition 6.15 (Cartesian-closed Categories)**.** It is said a category $\mathcal{K}$ is Cartesian-closed if it admits finite products and the functor $F = \_ \times a$ is a left adjoint for each $a \in \operatorname{Obj} \mathcal{K}$. Note $F(\alpha) : x \times a \to y \times a$.

Example 6.16.

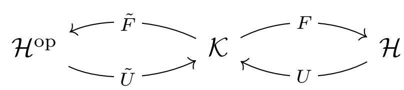
(1) $\mathcal{SET}$ is Cartesian-closed. For any given $a =: B \in \operatorname{Obj} \mathcal{SET}$, $F = \_ \times B$ is paired with $U = \mathcal{SET}(B, -)$. Then $F \dashv U$:

For $X, Y \in \operatorname{Obj} \mathcal{SET}$ we have $\mathcal{SET}(X \times B, Y) \simeq \mathcal{SET}(X, \mathcal{SET}(B, Y))$, as witnessed by $\phi_{X,Y} : Y^{X \times B} \simeq (Y^B)^X$.

(2) $\mathcal{POSET}$ is likewise Cartesian-closed. $\mathcal{RGRAPH}$, the category of reflexive graphs (i.e. graphs with loops), is likewise Cartesian-closed.

Definition 6.17 (Duality)**.** Let $F : \mathcal{K} \to \mathcal{H}$, $G : \mathcal{H} \to \mathcal{K}$. We are interested in $\mathcal{H}^{\text{op}}$ not $\mathcal{H}$. Instead of F, U we may consider the contravariant functors

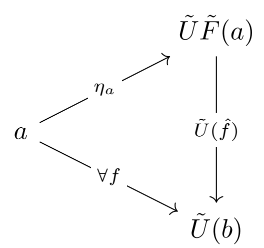
$$\tilde{F} : \mathcal{K} \to \mathcal{H}^{\text{op}} \qquad\qquad \tilde{U} : \mathcal{H}^{\text{op}} \to \mathcal{K}$$



Recall that for each $\alpha \in \mathcal{K}(a, a')$, Note $\tilde{F}(\alpha) = F(\alpha) \in \mathcal{H}^{\text{op}}(a', a)$ and $F(a) = \tilde{F}(a)$.

Have the unit and counit of adjunction changed? The unit $\eta : 1_{\mathcal{K}} \to UF = \tilde{U}\tilde{F}$ remains the same.

The counit is changed, however: Let $\{\epsilon_b \mid b \in \operatorname{Obj} \mathcal{H}\} = \epsilon : FU \to 1_{\mathcal{H}}$ be the original counit. Then the new counit $\{\tilde{\epsilon}_b \mid b \in \operatorname{Obj} \mathcal{H}^{\text{op}}\} = \tilde{\epsilon} : 1_{\mathcal{H}^{\text{op}}} \to \tilde{F}\tilde{U} (\neq FU)$ is given by $\epsilon_b = \tilde{\epsilon}_b$.



$\exists! \hat{f} \in \mathcal{H}^{\text{op}}(b, \tilde{F}(a))$

Example 6.18. Using the counit, verify $F \dashv U$. Hint: $\epsilon = (\epsilon_Y \mid FU(Y) \to Y)$ (note $FU(Y) = \mathcal{SET}(B, Y) \times B$), where $\epsilon_Y(f, b) = f(b)$, a so-called evaluation map.

Example 6.19. Let $\mathcal{B}$ be a category of complex Banach spaces and linear continuous maps between them. Let $B \in \operatorname{Obj} B$. Put $\tilde{B} = \{f \mid f \in \mathcal{B}(B, \mathbb{C})\}$ (where for $f \in \tilde{B}$ we put $\|f\| = \sup_{\|x\| \leqslant 1} \|f(x)\|$) is dual to B (a linear functional on B). Think how these act on morphisms through (exercise).

We obtain the unit $\eta = \{\eta_B \mid B \in \mathrm{Obj}\,\mathcal{H}\}$ such that $\eta_B : B \hookrightarrow \tilde{\tilde{B}}$ is an embedding. Note if we considered vectorspaces instead of Banach spaces, we would indeed get dual vectorspaces (i.e. spaces of linear forms on a vectorspace).

Example 6.20 (Stone Duality)**.** Boolean algebras and Boolean Spaces (compact, Hausdorff, totally disconnected topological spaces)

Example 6.21 (Priestly's Duality)**.** Priestly's duality is a duality between distributive $\{0, 1\}$-lattices and Priestly Spaces.

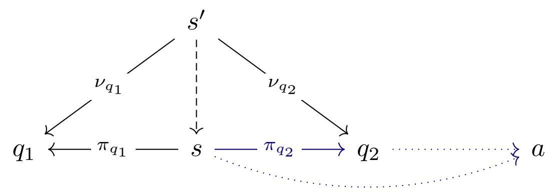
7 Adjoint Functors Theorem

Definition 7.1 (Quasi-Initial Objects)**.** Let $\mathcal{L}$ be a category, and $Q \subseteq \operatorname{Obj}\mathcal{L}$ be a set. It is said Q is a *quasi-initial* set of objects of $\mathcal{L}$ if for each $a \in \operatorname{Obj}\mathcal{L}$ there exists $q \in Q$ such that $\mathcal{L}(q, a) \neq \varnothing$.

The object $q \in \operatorname{Obj} Q$ is said to be *quasi-initial* if $\{q\}$ is a quasi-initial set (there is no requirement for uniqueness unlike with initial objects).

Lemma 7.2. *If $\mathcal{L}$ admits products and a quasi-initial set of objects then it admits a quasi-initial object.*

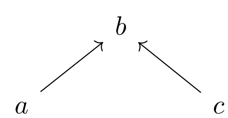
Proof. Let $Q \subseteq \operatorname{Obj}\mathcal{L}$ be quasi-initial. Define $s = \prod_{q \in Q} q$. Then s is a quasi-initial object.



QED

Example 7.3.

(1) A thin category defined as follows has a quasi-initial set $\{a, c\}$ but it does not have a quasi-initial object.



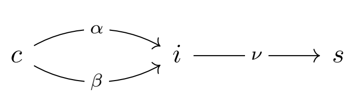
(2) Every small category $\mathcal{L}$ has a quasi-initial set of objects, namely $Q = \operatorname{Obj}\mathcal{L}$.

Lemma 7.4. *If $\mathcal{L}$ is a complete category with a quasi-initial set of objects, then $\mathcal{L}$ has an initial object.*

Proof. By Lemma 7.2, there is a quasi-initial object $s \in \operatorname{Obj}\mathcal{L}$. Let D be a full subcategory of $\mathcal{L}$ with $\operatorname{Obj} D = \{s\}$ (meaning $\operatorname{Mor} D = \mathcal{L}(s, s)$).

Denote $M : D \to \mathcal{L}$ a full embedding functor; this is a cone (D is small) which by assumption ($\mathcal{L}$ is complete) has a limit $\lim M = (i, \{\nu\})$, where $\nu : i \to s$ is a limit projection.

Then ν is a monomorphism

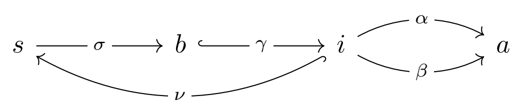


whence by definition $\nu \circ \alpha = \nu \circ \beta \Rightarrow \alpha = \beta$ (α, β are equal on a (unique) limit projection).

We claim i is the sought initial object (it clearly is quasi-initial since a morphism leads therefrom to s which is quasi-initial). Let $a \in \operatorname{Obj} \mathcal{L}$ be arbitrary, and $\alpha, \beta : i \to a$. We wish toshow $\alpha = \beta$. Consider $\gamma = \operatorname{eq}(\alpha, \beta) : b \to i$ (recall that all limits exist and, therefore, all equalisers do by Maranda's Theorem 4.19).

If we prove γ is an epimorphism, then $\alpha \circ \gamma = \beta \circ \gamma$ implies the claimed $\alpha = \beta$.

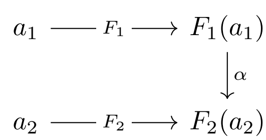
Let $\sigma \in \mathcal{L}(s, b)$ be arbitrary. Then $(\nu\gamma\sigma)\nu = \nu$ by definition of ν. Since ν is a monomorphism, $\nu(\gamma\sigma\nu) = \nu \circ 1_i \Rightarrow \gamma\sigma\nu = 1_i$ whence γ is an epimorphism (as it is a left inverse of the morphism $\sigma \circ \nu$ and therefore a retraction which is a particular case of epimorphisms).



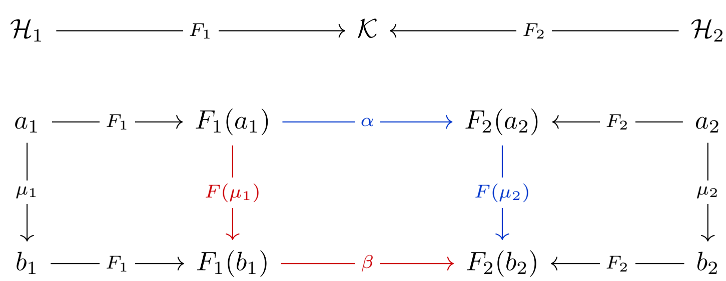
QED

Definition 7.5 (Comma Category)**.** Let $F : \mathcal{H}_1 \to \mathcal{K}$, $G : \mathcal{H}_2 \to \mathcal{K}$ be functors. We define the *comma category* $F_1 \downarrow F_2$:

Objects. The triple (a_1, α, a_2) is an object of $F_1 \downarrow F_2$ if $a_i \in \operatorname{Obj} \mathcal{H}_i$ for $i = 1, 2$ and $\alpha \in \mathcal{K}(F_1(a_1), F_2(a_2))$.

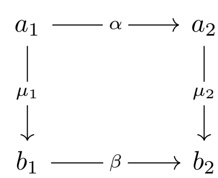


Morphisms. The pair (μ_1, μ_2) is a morphism in $F_1 \downarrow F_2$ from the objects (a_1, α, a_2) to (b_1, β, b_2) if $\mu_i : a_i \to b_i$ for $i = 1, 2$ with $F_2(\mu_2) \circ \alpha = \beta \circ F_1(\mu_1)$.



For $i = 1, 2$ we shall further denote by $P_i : F_1 \downarrow F_2 \to \mathcal{H}_i$ the left (for $i = 1$) and right ($i = 2$) projections; i.e. a functor with $P_i(a_1, \alpha, a_2) = a_i$, $P_i(\mu_1, \mu_2) = \mu_i$.

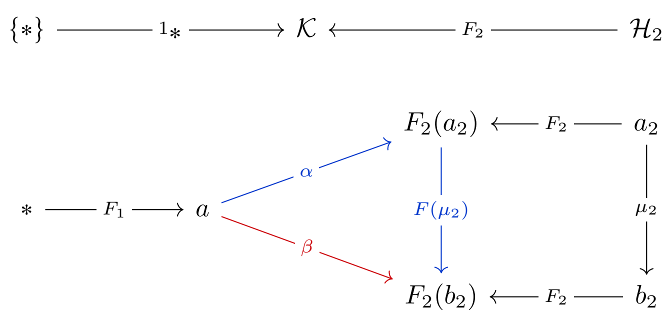
Example 7.6. Suppose $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{K}$, $F_1 = F_2 = 1_{\mathcal{K}}$. Then $F_1 \downarrow F_2$ has *de facto* morphisms of $\mathcal{K}$ as its objects and the pairs (μ_1, μ_2) for morphisms such that the following diagram dommutes. Then $F_1 \downarrow F_2 \simeq \mathcal{K}^D$ where D is of the form 'self-loop arrow self-loop'.



Notation 7.7. Let $\mathcal{H}_1 = 1$ be a discrete singleton category: $\operatorname{Obj} 1 = \{*\}$. with

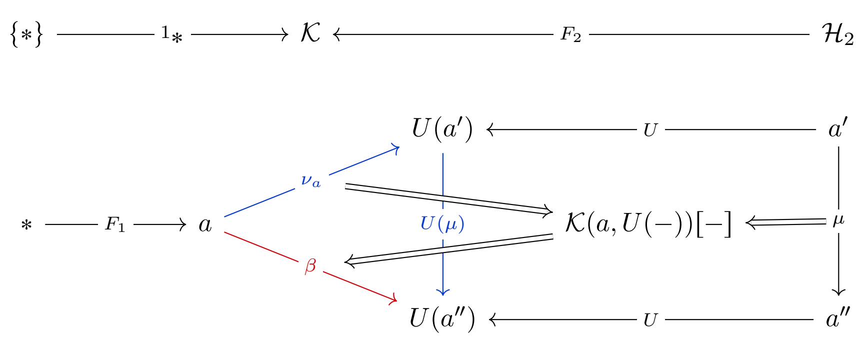
$$1 \xrightarrow{F_1} \mathcal{K} \xleftarrow{F_2} \mathcal{H}_2.$$

Set $a = F_1(*)$. Then instead of $F_1 \downarrow F_2$ we often simply write $a \downarrow F_2$, and instead of $(*, \alpha, a_2) \in \operatorname{Obj} a \downarrow F_2$ with $\alpha : a \to F_2(a)$; we also simply write $(\alpha, a_2) \in \operatorname{Obj} a \downarrow F_2$, and instead of $(1_*, \mu_2)$ simply μ_2).



Lemma 7.8 (Comma Category Observations)**.**

(1) Let $U : \mathcal{H} \to \mathcal{K}$, $a \in \operatorname{Obj} \mathcal{K}$. Then (a', ν_a) is a free object over a' with respect to U iff (ν_a, a') is an initial object of the category $a \downarrow U$.



(2) *Let $U : \mathcal{H} \to \mathcal{K}$ have a left adjoint. Then $F : \mathcal{K} \to \mathcal{H}$ iff the category $a \downarrow U$ has an initial object for every $a \in \operatorname{Obj} \mathcal{K}$. In such a case, this inintial object is of the form $(\nu_a, F(a))$ where ν is the unit of adjunction of $F \dashv U$.*

Example 7.9 (Important!)**.** Let D be a small category. We introduce:

(1) The category $\mathcal{H}^D$ defined as the category of all functors $M : D \to \mathcal{H}$.

(2) The functor $\Delta : \mathcal{H} \to \mathcal{H}^D$ termed the *constant functor* defined for $a \in \operatorname{Obj} \mathcal{H}$ by $\Delta(a) = \Delta_a : D \to \mathcal{H}$ (a constant functor on a) and for $\alpha \in \mathcal{H}(a, b)$ by $\Delta(\alpha) \in \operatorname{Nat}(\Delta_a, \Delta_b)$ such that $(\forall D \in \operatorname{Obj} d) \Delta(\alpha)_d = \alpha$.

For $M \in \operatorname{Obj} \mathcal{H}^D$ we consider the comma category $M \downarrow \Delta$ (M plays the role of a above). Then

$$(\nu, b) \in \operatorname{Obj} M \downarrow \Delta \Leftrightarrow (b, \nu) \text{ a cocone of the diagram } M.$$

Moreover, (ν, b) is an initial object in $M \downarrow \Delta$ iff $(b, \nu) = \operatorname{colim} M$ (exercise).

$$M \xrightarrow{\quad \nu \quad} \Delta_b$$

If $\mathcal{H}$ admits colimits of all diagrams with the scheme D, then Lemma 7.8 (2) proves there exists a left adjoint F of Δ and $\operatorname{colim} M = (F(b), \nu_b)$, where ν is a unit of adjunction of $F \dashv \Delta$.

Theorem 7.10 (Comma Category Completeness)**.** *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be complete categories $F_i : \mathcal{H}_i \to \mathcal{K}$ for $i = 1, 2$. If F_2 preserves limits, then $F_1 \downarrow F_2$ is complete.*

Proof. Received in paper form. QED

Definition 7.11 (Solution-Set-Condition)**.** Let $U : \mathcal{H} \to \mathcal{K}$ be a functor. It is said U satisfies the *Solution-Set-Condition* or *SSC* if for every $a \in \operatorname{Obj} \mathcal{K}$ the category $a \downarrow U$ admits a quasi-initial set of objects.

Theorem 7.12 (Adjoint Functor Theorem)**.** *Let $U : \mathcal{H} \to \mathcal{K}$ be a functor, and $\mathcal{H}$ be complete. Then U has a left adjoint iff U preserves limits and satisfies SSC.*

Proof.

($\Rightarrow$) We know the right adjoint preserves limits. By Lemma 7.8 (2), $a \downarrow U$ admits an initial object for each $a \in \operatorname{Obj} \mathcal{K}$, whence SSC is satisfied.

($\Leftarrow$) By Lemma 7.8 (2), it suffices to show $(\forall a \in \operatorname{Obj} \mathcal{K})$ $a \downarrow$ admits an initial object. By Theorem 7.10, $a \downarrow U$ is complete.

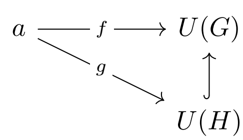
By Lemma 7.4 and SSC, $a \downarrow U$ admits an initial object for each $a \in \operatorname{Obj} \mathcal{K}$.

QED

Example 7.13 (An Application: Existence of Free Groups)**.** Consider the forgetful functor $U : \mathcal{GRP} \to \mathcal{SET}$ and permissibly $F : \mathcal{SET} \to \mathcal{GRP}$ (we do not know if it exists).

If we do not know $\mathcal{GRP}$ admits coproducts $\coprod_{x \in a} \mathbb{Z}$ for any a (which would imply the existence of F), we may employ Adjoint Functor Theorem.

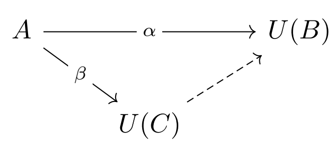
To verify SSC, $a \in \operatorname{Obj} \mathcal{SET}$, $G \in \operatorname{Obj} \mathcal{GRP}$. For the quasi-initial set, we take the set $Q = \{(g, H) \mid g \in \mathcal{SET}(a, U(H)), H \in \mathcal{S}\}$ where $\mathcal{S}$ is a representative set of groups generated by fewer than $|a|$ elements.



7.1 Topological Applications

Let $\kappa = \mathcal{TOP}$, $\mathcal{H} \in \{\mathcal{HAUS} \mathcal{CHAUS}\}$. Let $U : \mathcal{H} \hookrightarrow \mathcal{K}$ be a full embedding functor. U admits a left adjoint F iff $\mathcal{H}$ is a reflexive subcategory. Note $\mathcal{H}$ is a complete category (its limits are the same as in $\mathcal{TOP}$ and U preserves limits. Lastly, SSC is met: Let We need to check $A \in \operatorname{Obj} \mathcal{K}$ $A \downarrow U$ has a quasi-initial set of objects.

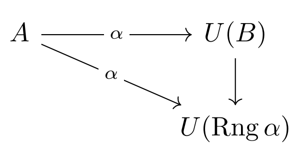
$(\alpha, B) \in \operatorname{Obj} A \downarrow U$



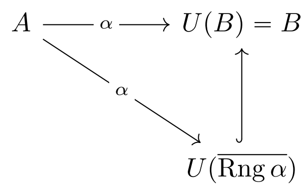
For the quasi-initial set, take

$\mathcal{H} = \mathcal{HAUS}$ $Q = \{(\beta, C) \mid \beta : A \to U(C), C \in \mathcal{S}\}$, where $\mathcal{S}$ is a representative set of objects in $\mathcal{H}$ with cardinality not greater than $|A|$.

$\operatorname{Rng} \alpha \leqslant U(B) = B$ with $|\operatorname{Rng} \alpha| \leqslant |A|$.



$\mathcal{H} = \mathcal{CHAUS}$ $Q = \{(\beta, C) \mid \beta : A \to U(C), C \in \mathcal{T}\}$, where $C \in T$ is a representative set of objects from $\mathcal{H} = \mathcal{CHAUS}$ with the cardinality $|C| \leqslant 2^{2^{|A|}}$.



Overall, there exists an F such that $F \dashv U$ (where F is 'Hausdorffied' if $\mathcal{H} = \mathcal{HAUS}$ and compactified if $\mathcal{H} = \mathcal{CHAUS}$).

Example 7.14. SSC may not be ignored. One may take $\mathcal{H}$ as the opposite category of the (thin) category of ordinal numbers. $\mathcal{H}$ is complete (but it is not cocomplete as it has no initial object).

Let $F : \mathcal{SET} \to \mathcal{H}$, $U : \mathcal{H} \to \mathcal{SET}$. U is representable and therefore it preserves limits but there is no left adjoint of F (which preserves colimits) to U, since $F(\varnothing)$ would need to be an initial object (recall $\varnothing$ is an initial object in $\mathcal{SET}$).

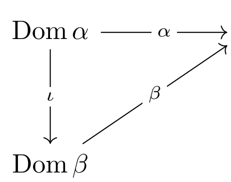
Theorem 7.15 (Dual Form of AFT)**.** *Let $F : \mathcal{K} \to \mathcal{H}$ be a functor, $\mathcal{K}$ be cocomplete. Then F has a right adjoint iff F preserves colimits and does not satisfy 'co-SSC'; i.e. $(\forall b \in \operatorname{Obj}\mathcal{H})\, F \downarrow b$ has a quasi-initial set of objects.*

Exercise 7.16. For the functor $\Delta : \mathcal{H} \to \mathcal{H}^D$, where D is small, $\Delta(a) := \Delta_a := \Delta_a : D \to \mathcal{H}$ is constant (and $\Delta(\alpha)_d := \alpha$). Show that Δ preserves limits and colimits.

An Application. $\mathcal{H}$ is complete and Δ has a left adjoint of F iff $\mathcal{H}$ has colimits of all diagrams from the scheme D. From AFT we know that for the existence of colimits it suffices to shwo that for every $M \in \operatorname{Obj}\mathcal{H}^D$ (i.e. a diagram) there exists a quasi-initial set of objects from $M \downarrow \Delta$ (e.g. (ν, b)); i.e. a cocone (b, ν).

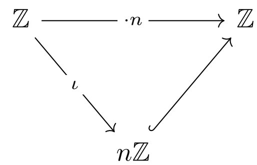
8 Subobjects, Factorobjects, & SAFT

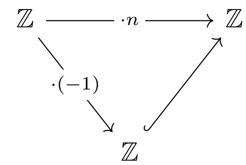
Definition 8.1 (Subobject)**.** Let $\mathcal{K}$ be a category and $a \in \operatorname{Obj}\mathcal{K}$. Every monomorphism $\alpha \in \mathcal{K}(p, a)$ is termed an *subobject* of a. Two subojects α, β of a are identified if there exists an isomorphism $\iota \in \mathcal{K}(\operatorname{Dom}\alpha, \operatorname{Dom}\beta)$ such that $\beta \circ \iota = \alpha$.



Dually, a factorobject of a is an epimorphism $\gamma \in \mathcal{K}(a, b)$ construed under the same condition for sameness as above.

Example 8.2. In $\mathcal{AB}$ (or $\mathcal{GRP}$) take $a = (\mathbb{Z}, +, -, 0)$. Every pair of distinct objects is of the form $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$, $n \in \mathbb{N} \cup \{0\}$.





Definition 8.3 (Wellpowered Category)**.** A category $\mathcal{K}$ is termed *wellpowered* if each object therein admits merely a set of pairwise distinct subobjects.

Dually, a category is *co-wellpowered* if each such object admits a set of factorobjects.

The name is indicative (perhaps) of $\mathcal{SET}$ satisfying the Axiom of Powerset.

Example 8.4.

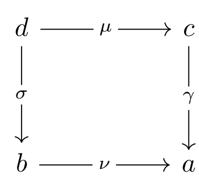
(1) The thin category of ordinals is wellpowered but is not co-wellpowered.

(2) A category $\mathcal{K}$ of Urysohn Topological Spaces ('every two points may be separated by two disjoint closed neighbourhoods') is wellpowered but is not co-wellpowered (J. Schröder, 1983).

Lemma 8.5.

(1) Let $\mathcal{K}$ admit products and a cogenerator $c \in \operatorname{Obj}\mathcal{K}$. Then for each $b \in \operatorname{Obj}\mathcal{K}$ there is a set Y and a monomorphism $\nu : b \to c^Y (= \prod_{y \in Y} c)$.

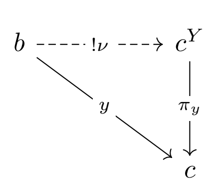
(2) If the following diagram is a pullback in $\mathcal{K}$ and ν is a monomorphism, then μ is a monomorphism.

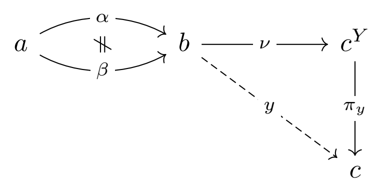


Proof.

(1) For each fixed $b \in \operatorname{Obj}\mathcal{K}$, set $Y := \mathcal{K}(b, c)$. Since our categories are (by agreement) locally small, Y is indeed a set.

Define the morphism ν on projections π_y (they are all the same) of the product c^Y by $\pi_y \circ \nu = y$. Then (b, Y) is a cone of the product diagram.

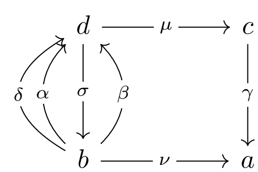




Then ν is a monomorphism; let $\alpha, \beta : a \to b$ be distinct. We shall show this implies $\nu \circ \alpha, \nu \circ \beta$ are likewise distinct.

Recall c is a cogenerator, whence there exists some $y \in Y$ such that $y \circ \alpha \neq y \circ \beta$ and therefore, indeed, $\nu \circ \alpha \neq \nu \circ \beta$.

(2) Let us have $\alpha, \beta : b \to d$ such that $\mu \circ \alpha = \mu \circ \beta$. We are showing $\alpha = \beta$. We know that $\gamma \circ \mu \circ \alpha = \gamma \circ \mu \circ \beta$, i.e. $\nu \circ \sigma \circ \alpha = \nu \circ \sigma \circ \beta$. Since ν is a monomorphism, we have $\sigma \circ \alpha = \sigma \circ \beta$. Hence $(l, \sigma \circ \alpha, \mu \circ \alpha, \gamma\mu\alpha))$ is a cone of the pullback diagram, whence there exists a unique $\delta : b \to d$ such that $\mu \circ \delta = \mu \circ \alpha = \mu \circ \beta$ and $\sigma \circ \delta = \sigma \circ \alpha = \sigma \circ \beta$, this altogether implies $\delta = \alpha = \beta$.



QED

Theorem 8.6 (Special Adjoint Functors Theorem (SAFT))**.** *Let $\mathcal{H}$ be a complete category which is wellpowered with a cogenerator. Then $U : \mathcal{H} \to \mathcal{K}$ admits a left adjoint iff U preserves limits.*

Proof.

$\Rightarrow$ Corollary 6.7.1

$\Leftarrow$ The chief strategy is to use AFT: we need to show SSC holds to obtain RHS.

Fix $a \in \operatorname{Obj} \mathcal{K}$. We need to show $a \downarrow U$ admits a quasi-initial set of objects. Set $X = \mathcal{K}(a, U(c))$.

Furthermore, let (c^X, π) be a product ($c^X = \prod_{\gamma \in X} c$ and π are projections). Denote by M a (representative) set of subobjects of c^X (recall $\mathcal{H}$ wellpowered and hence it is indeed a set). Namely,

$$M \subseteq \left\{ \mu : \operatorname{Dom}(\mu) \to c^X \,\middle|\, \mu \text{ is a monomorphism} \right\}.$$

Put

$$\operatorname{Obj} a \downarrow U \supseteq Q = \{(\alpha, \operatorname{Dom}\mu)) \mid \mu \in M, \alpha \in \mathcal{K}(a, U(\operatorname{Dom} *(\mu)))\}\,.$$

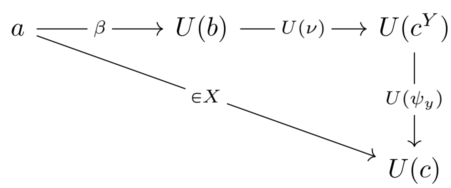
Henceforth, we shall write $d_\mu := \operatorname{Dom}\mu$. We claim Q is a quasi-initial set in $a \downarrow U$.

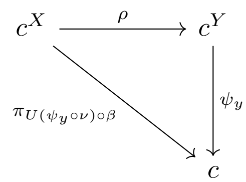
Let $(\beta, b) \in \operatorname{Obj} a \downarrow U$ be any fixed object. We are trying to find some morphism leading from some $(\alpha, \operatorname{Dom}\mu)$ to (β, b). We have a lemma at hand, which we shall use to insert β into some set of cogenerators of c.

By 8.5 (1) there exists a set Y with a monomorphism $\nu : b \to c^Y$. Denote by (c^Y, ψ) the corresponding product (whose existence we assume from the outset).

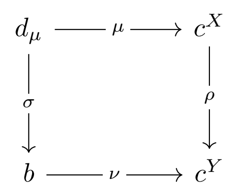
Define $\rho : c^X \to c^Y$ on the projections of ψ by

$$(\forall y \in y)\, \psi_y \circ \rho = \pi_{U(\psi_y \circ \nu) \circ \beta (\in X)}$$



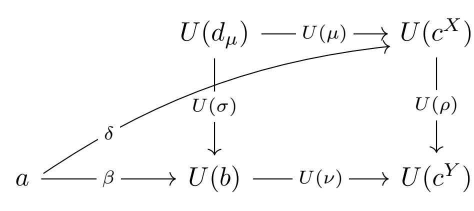


We construct a pullback



Note the top-left corner of a pullback is uniquely determined up to isomorphism: consequently from those isomorphic choices we may in particular choose d_μ (note μ is a monomorphism by Lemma 8.5(2).

We apply the functor U (preserving limits) and obtain the new pullback



U preserves products, whence $(U(c^X), U(\pi))$ forms a product. We define $\delta : a \to U(c^X)$ on the projections of $U(\pi)$ by

$$(\forall \gamma \in X)\, U_{(\pi_\gamma)\circ\delta} = \gamma$$

We claim $(a, (\beta, \delta U(\rho) \circ \delta \overset{=}{?} U(\nu) \circ \beta))$ is a cone of the pullback diagram. We, therefore, need to verify $U(\rho) \circ \delta = U(\nu) \circ \beta$: "a since na projekcich U_ψ soucinem $(U(c^Y), U(\psi)$.

$$\begin{aligned} U(\psi_y) \circ U(\nu) \circ \beta &= U(\psi_y \circ \nu) \circ \beta \\ &\overset{\text{Def of } \delta}{=} U(\pi_{U(\psi_y \circ \nu) \circ \beta}) \circ \delta \\ &\overset{\text{Def of } \rho}{=} U(\psi_y \circ \rho) \circ \delta \\ &= U(\psi_y) \circ U(\rho) \circ \delta. \end{aligned}$$

We have verified: $U(\nu) \circ \beta = U(\rho) \circ \delta$. By the universal property of pullbacks, there exists $\alpha : a \to U(d_\mu)$ such that $\beta = \alpha \circ U(\sigma)$. That is to say,

$$\sigma \in a \downarrow U(\underbrace{(\alpha, d_\mu)}_{\in Q}, (\beta, b)).$$

Strategy Overview. The most important step is to define ρ. We want to use the universal property of pullbacks, and hence the second important step is defintion of δ. A verification follows. Finally, find α such that $\beta = U(\sigma) \circ \alpha$. σ had not played any significant role until when when it is shown to witness the claim. QED

Corollary 8.6.1. *Suppose $\mathcal{H}$ is complete, wellpowered, and admits a cogenerator, then $\mathcal{H}$ is also cocomplete.*

Proof. Let $M : D \to \mathcal{H}$ be a diagram. Recall $\mathcal{H} \overset{[}{\to} \Delta]\mathcal{H}^D$ preserves limits. It follows from SAFT that there exists a left adjoints F to Δ, then $\operatorname{colim} M = (F(M), \nu_M)$ where μ is a unit of adjunction $F \dashv \Delta$.

Specifically $\mathcal{CHAUS}$ is cocomplete (as it is wellpowered and complete). Its cogenerator is the closed interval $[0, 1]$ with the classic topology. QED

Example 8.7.

(1) The requirement $\mathcal{H}$ be well-powered may not be omitted: Let $\mathcal{H}$ be the opposite category of the thin category of ordinal numbers. Note $\mathcal{K}$ is complete, admits a cogenerator but it is not well-powered (and does not have an initial object).

The representable functor $U : \mathcal{H} \to \mathcal{SET}$ cannot admit a left adjoint. It follows then the requirement on wellpoweredness is essential.

(2) The requirement $\mathcal{H}$ admit a cogenerator cannot likewise be omitted.

(a) The category $\mathcal{CLAT}$ of complete lattices endowed with maps preserving meets and joins is complete and well-powered, but it does not admit coproducts. That is, the functor $U := \Delta : \mathcal{H} \to \mathcal{H}^\Delta$ preserves limits, it may not admit a left adjoint F if D is a discrete category such that $\mathcal{H}$ does not admit a coproduct of some $M : D \to \mathcal{H}$.

In particular, the following lattice

$$0 \longrightarrow x \longrightarrow 1$$

does not admit a coproduct of three copies of

$$0 \longrightarrow x \longrightarrow 1$$

(b) Let $U : \mathcal{GRP} \to \mathcal{SET}$ be a functor defined as follows:

i. For every infinite cardinal λ, thereeists a simple group A_λ of cardinality λ.

ii. For every ordinal number α we set $A_\alpha := A_{|\alpha|}$.

We define $U(\boldsymbol{G}) = \prod_{\omega \leqslant \alpha \in \mathrm{O_n}, |\alpha| \leqslant |G|} \mathcal{GRP}(A_\alpha, G) = \mathcal{GRP}\left(\coprod^*_{\omega \leqslant \alpha \in \mathrm{O_n}, |\alpha| \leqslant |G|} A_\alpha, G\right)$.

Think about how U is defined on morphisms (similarly to a covariant homfunctor). Then U preserves limits (this follows from 'more-or-less' from the fact covariant homfunctors preserve limits) but is is not representable (by contradiction, a proposition from set theory), whence it does not admit a left adjoint.

(3) To show the final counterexample, it shall suffice to find some $\mathcal{H}$ which admits a cogenerator, is well-powered, is not cocomplete. If we find it, we can re-employ the strategy from (2) (and consider some suitable $U := \Delta : \mathcal{H} \to \mathcal{H}^D$.

Consider then the wellpowered small category which is not thin (and consequently by Freyd's Theorem is neither complete nor cocomplete) with a cogenerator: the group $\mathbb{Z}_2$ thoguht of as a singleton category: $\mathrm{Obj}\,\mathcal{K} = \{*\}$ and $\mathrm{Mor}\,\mathcal{K} = \{1_*, \alpha\}$. Then $*$ is a cogenerator.

$$* \underset{\beta}{\overset{\alpha}{\rightrightarrows}} * \xrightarrow{\;1_*\;} * \qquad (\alpha \neq \beta)$$

Exercise 8.8. Let $\mathcal{K}$ be a category, $\pi \in \mathcal{K}(a,b)$ be an epimorphism and $c \in \mathrm{Obj}\,\mathcal{K}$. Then the following is a monomorphism in $\mathcal{SET}$.

$$\mathcal{K}(\pi, c) : \mathcal{K}(b,c) \to \mathcal{K}(a,c)$$

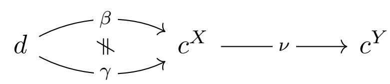
Lemma 8.9. *Let $\mathcal{K}$ admit products of the form c^X for a fixed $c \in \mathrm{Obj}\,\mathcal{K}$ and any set X.*

If $\alpha : X \to Y$ is a monomorphism in $\mathcal{SET}$ and $X \neq \varnothing$ then there exists some monomorphism $\nu : c^X \to c^Y$ in $\mathcal{K}$.

Proof. Denote the products in question $(c^X, \pi), (c^Y, \psi)$. The morphism ν is defined on the projections ψ by

$$(\forall y \in Y)\, \psi_y \circ \nu = \begin{cases} \pi_x & \text{if} \quad \alpha(x) = y. \\ \pi_p & \text{else} \qquad \text{for some fixed } p \in X. \end{cases}$$

We shall verify it is a monomorphism: i.e. we wish to show $\beta \neq \gamma \Rightarrow \nu \circ \beta \neq \nu \circ \gamma$.



Since $\beta \neq \gamma$,

$$(\exists x \in X)\, \pi_x \circ \beta \neq \pi_x \circ \gamma,$$

then

$$\underbrace{\psi_{\alpha(x)} \circ \nu}_{\pi_x} \circ \beta \neq \underbrace{\psi_{\alpha(x)} \circ \nu}_{\pi_x} \circ \gamma \Rightarrow \nu \circ \beta \neq \nu \circ \gamma.$$

QED

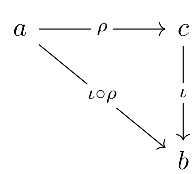
Theorem 8.10. *A category $\mathcal{K}$ is wellcopowered iff*

$(\forall a \in \operatorname{Obj} \mathcal{K}\, \exists M \subseteq \operatorname{Obj} \mathcal{K})\, M$ *is a set* $\wedge$ $(\forall$ *factorobject* π *of* $a\, \exists b \in \operatorname{Obj} \mathcal{K})\ \operatorname{Cod} \pi \simeq b$.

Proof.

- $\Rightarrow$ Trivially by definition.
- $\Leftarrow$ Let $a \in \operatorname{Obj} \mathcal{K}$ be fixed, $\pi : a \to b$ be a factorobject. It suffices to show factorobjects $\rho : a \to c$ where $c \simeq b$ form a set.

  If $\rho : a \to c$ is some factorobject, we fix some isomorphism $\iota : c \to b$ (whose existence we assume). Then ρ is identical with $\iota \circ \rho$ with respect to the equivalence defined on factorobjects.

  Without any loss of generality, assume $\rho : a \to b$. But $\mathcal{K}(a, b)$ is merely a set, whence factorobjects of a with $\operatorname{Cod}(\_) \simeq b$ form a set.

QED

Theorem 8.11. *Let $\mathcal{K}$ be complete, wellpowered, and admit a cogenerator $c \in \operatorname{Obj} \mathcal{K}$. Then $\mathcal{K}$ is wellcopowered.*

Proof. We will make use of Theorem 8.10. Let $a \in \operatorname{Obj} \mathcal{K}$ be fixed and $\pi : a \to b$ be a factorobject.

Lemma 8.5 implies there exists a monomorphism $\mu : b \to c^X$ with $X = \mathcal{K}(b, c)$ (the latter part follows from the construction in the proof thereof).

If $X \neq \varnothing$, consider the injective map $\alpha = \mathcal{K}(\pi, c) : X \to \mathcal{K}(a, c) =: Y$. By Lemma 8.9 we obtain $\nu : c^X \to c^Y$ is a monomorphism.

Altogether then, we have a monomorphism $\nu : \circ\mu : b \to c^Y$, where $Y = \mathcal{K}(a, c)$ is independent of b.

We have shown the codomain of any factorobject a is the domain of a subobject of c^Y or the domain of the terminal object (which was the case $X = \varnothing$). Since $\mathcal{K}$ is wellpowered, the proof is concluded. QED

Remark 8.12. The proof would work just as well if $\mathcal{K}$ simply admitted products.

Example 8.13.

(1) The assumption about the existence of a cogenerator may not be omitted. Take $\mathcal{K}$ as the category of Urysohn topological spaces: this category is complete, wellpowered but it does not admit a cogenerator and is not wellcopowered.

(2) Let $\mathcal{K}$ be the thin category of ordinal numbers with an added greatest element.

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow \omega_0 \longrightarrow \cdots \longrightarrow \infty$$

$\alpha \in \mathrm{O_n}$ admits a proper class of factorobjects, and ∞ admits a proper class of subobjects.

9 Dense & Colimit-Dense Categories

Definition 9.1 (Colimit-Dense Category, Canonical Diagram)**.** Let $\mathcal{K}$ be a category, and $\mathcal{H} \subseteq \mathcal{K}$ be a full subcategory, $G : \mathcal{H} \to \mathcal{K}$ be an embedding functor.

(1) $\mathcal{H}$ is said to be a colimit-dense subcategory in $\mathcal{K}$ if $(\forall a \in \operatorname{Obj} \mathcal{K})$ there exists a diagram $M : D \to \mathcal{H}$ such that $(a, \nu) = \operatorname{colim}(G \circ M)$ for a suitable ν.

(2) If $\mathcal{H}$ small, consider the (small) comma-category $G \downarrow a$ for some $a \in \operatorname{Obj} \mathcal{K}$.

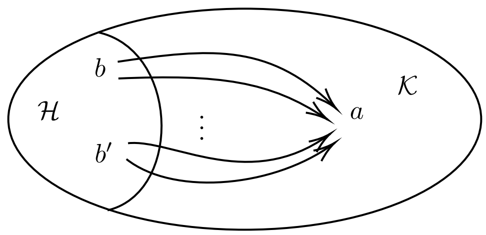
The functor $K : G \downarrow a \to \mathcal{K}$ defined on objects $(b, \alpha) \in \operatorname{Obj} G \downarrow a$ and morphisms $\mu \in \operatorname{Mor} G \downarrow a$ by

$$\mathcal{K}(b, \alpha) = b = G(b) \qquad\qquad \mathcal{K}(\mu) = \mu = G(\mu)$$

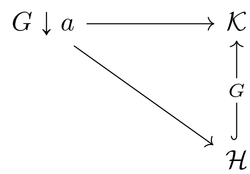
is said to be the *canonical diagram* of a with respect to $\mathcal{H}$.

(3) It is said $\mathcal{H}$ is a (small) dense subcategory of $\mathcal{K}$ if $(\forall a \in \operatorname{Obj}\mathcal{K})$ is $\operatorname{colim} K = (a, \{\alpha \mid (b,\alpha) \in \operatorname{Obj} G \downarrow a\})$

If K is the canonical diagram of the object a with respect to $\mathcal{H}$, then $(a, (\alpha, (b,\alpha) \in \operatorname{Obj} G \downarrow a))$ is always a cocone of the diagram K.



$\mathcal{H}$ is dense by definition iff $(\forall a \in \operatorname{Obj}\mathcal{K})(a, (\alpha, (b,\alpha) \in \operatorname{Obj} G \downarrow a)) = \operatorname{colim} K$.



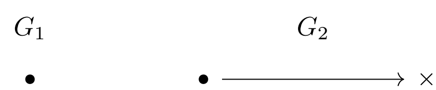
Remark 9.2. A dense subcategory is colimit-dense.

Example 9.3.

(1) In $\mathcal{SET}$, $\mathcal{H}$ with a single object $\{*\} \in \operatorname{Obj}\mathcal{H}$ is a small, dense subcategory.

$$a = \coprod_{x \in a} \{*\}\,.$$

(2) In $\mathcal{GRAPH}$ (the category of oriented graphs) there does not exists a singleton dense subcategory but $\mathcal{H}$, where $\operatorname{Obj}\mathcal{H} = \{G_1, G_2\}$ where G_1, G_2 are defined as follows is



and therefore $G_1 = \{\{\bullet\}, \varnothing\}$ and $G_2 = \{\{\bullet, \times\}, (\bullet, \times)\}$ Our G_1, G_2 play the role of b, b' from earlier. Note there exists an arrow from G_1 to every vertex of any graph a (a per illustration above) while G_2 has an arrow leading to any two vertices connected with an edge (??).

(3) Let $\boldsymbol{T}$ be a field. Let $\mathcal{K} = \operatorname{Mod} - \boldsymbol{T}$ the category of vectorspaces over $\boldsymbol{T}$. Note $\mathcal{H}$ where $\operatorname{Obj}\mathcal{H} = \{\boldsymbol{T}\}$ is a colimit-dense subcategory in $\mathcal{K}$ (every vectorspace $\boldsymbol{V}$ is isomorphisc to $\bigoplus_{i\in} \boldsymbol{T}$).

This category is not a dense subcategory in $\mathcal{K}$. Conversely, $\mathcal{H}'$ wheree $\operatorname{Obj}\mathcal{H}' = \{\boldsymbol{T} \times \boldsymbol{T}\}$. is a dense subcategory.

Generally, If $\mathcal{K}$ is a variety of univerfsal algebras with a signature in which all operations have an arity at most $n \in \mathbb{N}$, then the free algebra (in this variety) with n generators forms a dense singleton subcategory.

(4) For $\mathcal{K} = \mathcal{TOP}$, the subcategory $\mathcal{H}$ composed of all Hausdorff and totally disconnected spaces is colimit dense. This $\mathcal{H}$ is not a small subcategory.

Exercise 9.4. The forgetful functor $F : \mathcal{TOP} \to \mathcal{SET}$ admits both right and left adjoint.

Solution.

$$\mathcal{TOP} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{SET}$$

Define $U(X) = (X, \tau)$, where $\tau = \{\varnothing, X\}$; i.e. the indiscrete topology.

The Unit:

$$\eta_{Y,\sigma} = \mathrm{Id}_Y : (Y, \sigma) \to \underbrace{(Y, \{\varnothing, Y\})}_{UF((Y,\sigma))}.$$

The rest is left as an exercise. QEF

Remark 9.5. The furgetful functor $\mathcal{TOP} \to \mathcal{SET}$ preserves colimits (and limits).

Theorem 9.6. *$\mathcal{TOP}$ admits no small colimit-dense subcategory*

Proof. Towards a contradiction, suppose $\mathcal{H} \subseteq \mathcal{K}$ is a small, colimit-dense subcategory in $\mathcal{TOP}$. Denote the forgetful functor $F : \mathcal{TOP} \to \mathcal{SET}$. If $\mathcal{H}$ is small, we may consider some set which has larger cardinality than all those in $\mathcal{H}$.

Let X be a set of larger cardinality than every $F(b)$ where $b \in \mathrm{Obj}\,\mathcal{H}$. Assume X is at least infinite.

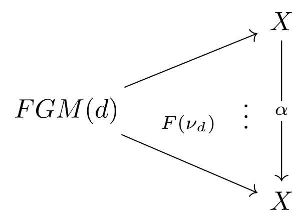
We define on X a topology (verify it is a topology)

$$\rho := \{P \subseteq X \mid |X \backslash P| < |X|\} \cup \varnothing.$$

If $Y \subseteq X$, $|Y| < |X|$, then the subspace (Y, ρ) is discrete. For any $y \in Y$, consider $P_y = \{y\} \cup X \backslash Y$ — an open set in X, whence it is open in Y.

Suppose $\mathcal{H}$ is colimit-dense. Then there exists a diagram $M : D \to \mathcal{H}$ such that $(X, \rho) = \mathrm{colim}(G \circ M)$, where $G : \mathcal{H} \to \mathcal{K}$ is an embedding. Then $X = ((X, \rho)\nu) = \mathrm{colim}(F \circ G \circ M)$ (for some nu), and therefore 1_x is the only map

$\alpha : X \to X$ such that $(\forall d \in \operatorname{Obj} D)\, \alpha \circ F(\nu_d) = F(\nu_d)$.



Consider (X, σ) where σ is a discrete topology. Then $((X, \sigma), \nu)$ is a cocone of the diagram $G \circ M$ (i.e. $\nu_d : GM(d) \to (X, \sigma)$ are continuous; since $\nu_d : GM(d) \to (X, \rho)$ is continuous and by our choice of (X, ρ) (??))

$(\forall d \in D)\alpha \circ \nu_d = \nu_d$ implies by definition of α taht $\alpha = 1_X$ but 1_X is not continuous, since (X, ρ) is not discrete). QED

Definition 9.7. Let $\mathcal{H} \subseteq \mathcal{K}$ be a small, full subcategory. We define the canonical functor (for $\mathcal{H}$) by:

$$H : \mathcal{K} \to \mathcal{SET}^{(\mathcal{H}^{\text{op}}}$$
$$(\forall c \in \operatorname{Obj} \mathcal{K})\, c \mapsto \mathcal{K}(\_, c) \restriction \mathcal{H}^{\text{op}}$$
$$(\delta : c \to d)\, \delta = \mathcal{K}(\_, \delta) \restriction \mathcal{H}^{\text{op}}$$
$$(\forall b \in]Obj\mathcal{H}^{\text{op}}\, (H(\delta))_b = \mathcal{K}(b, \delta) : \mathcal{K}(b, c) \to \mathcal{K}(b, d).$$

Theorem 9.8. *The subcategory $\mathcal{H} \subseteq \mathcal{K}$ is small, full, and dense iff the canonical functor of H is faithful and full.*

Proof. Next time. QED

Corollary 9.8.1. *If $\mathcal{K}$ admits a dense subcategory $\mathcal{H}$, then $\mathcal{K}$ is equivalent to (its image) some full subcategory in $\mathcal{SET}^{\mathcal{H}^{\text{op}}}$.*

Theorem 9.9. *Let $\mathcal{H}$ be a small and complete subcategory of $\mathcal{K}$. Then $\mathcal{H}$ is dense iff the canonical functor $H : \mathcal{K} \to \mathcal{SET}^{\mathcal{K}^{\text{op}}}$ is full and faithful.*

Proof. Let $c, d \in \operatorname{Obj} \mathcal{K}$ be arbitrary. Let $\tau : H(c) \to H(d)$; i.e. $\tau \in \operatorname{Nat}(H(c), H(d))$. Compatibility conditions on τ immediately yield $(d, (\tau_b(\gamma))$ where $(b, \gamma) \in \operatorname{Obj} G \downarrow c))$[4] is a cocone of the canonical diagram $K : G \downarrow c \to \operatorname{Obj} c$ with respect to the subcategory $\mathcal{H}$; here G is an embedding functor.

Conversely, if (d, ν) is a cocone of the canonical diagram $K : G \downarrow c \to \mathcal{K}$ of the object c, then $\tau(\tau_b, b \in \operatorname{Obj} \mathcal{H}^{op})$, where τ is defined element-wise by $\tau_b(\gamma) = \nu_{(b,\gamma)}$ where $b, \gamma \in \operatorname{Obj} G \downarrow c$, is a natural transformation from $H(c)$ into $H(d)$.

[4] where $\gamma \in \mathcal{K}(b, c)$.

H is full and faithful whence, for any $\tau \in \mathrm{Nat}(H(c), H(d))$, there exists a unique $\delta : c \to d$ such that for each $b \in \mathrm{Obj}\,\mathcal{H}^{\mathrm{op}}$, $\tau_b = \mathcal{K}(b, \delta)$.

$\mathcal{H}$ is dense, meaning $(c, (\gamma, (b, \gamma) \in \mathrm{Obj}\,G \downarrow c)) = \mathrm{colim}\,K$, there exists a unique $\delta : c \to d$ such that for each $(b, \gamma) \in \mathrm{Obj}\,G \downarrow c)$, $\nu_{(b,\gamma)} = \delta \circ \gamma = \tau_b(\gamma) = \mathcal{K}(b, \delta)(\gamma)$. QED

Example 9.10. Now that we have acquired new tools, let us reexamine **??**. Let $\mathcal{K}$ be the category of vectorspaces over $\boldsymbol{T}$. $\mathcal{H}$, where $\mathrm{Obj}\,\mathcal{H} = \{\boldsymbol{T}\}$ is not dense (although it is colimit-dense) in $\mathcal{K}$.

Let $\boldsymbol{V} = \boldsymbol{T}^2 \in \mathrm{Obj}\,\mathcal{K}$, $\boldsymbol{W} = \boldsymbol{T}$ and $g \colon \boldsymbol{V} \to \boldsymbol{W}$ is a map which is not additive, that is $g \notin \mathrm{Mor}\,\mathcal{K}$, but $(\forall t \in T)(\forall v \in V)\, g(vt) = g(v)t$ (which may be easily verified/constructed).

$\tau : H(V) \to H(W)$ is defined on (its only) component thus:

$$\tau_{\boldsymbol{T}} : \mathcal{K}(\boldsymbol{T}, \boldsymbol{V}) \to \mathcal{K}(\boldsymbol{T}, \boldsymbol{W})$$
$$f \mapsto g \circ f$$

This indeed is additive[5] and thus a linear map. τ is a natural transformation, but there edoes not exist any $\delta : \boldsymbol{V} \to \boldsymbol{W}$ is linear such that $\tau = H(\delta) : g : \boldsymbol{V} \to \boldsymbol{W}$ is the only option for δ but g is not linear.

Remark 9.11 (Isbell)**.** $\mathcal{SET}^{\mathrm{op}}$ admits a small dense subcategory iff there does not exist a proper class of measurable cardinals.

Theorem 9.12 (CMUC Magazine 2019)**.** *$\mathcal{SET}^{\mathrm{op}}$ admits a subcategory $\mathcal{H}$ where* $\mathrm{Obj}\,\mathcal{H} = (\{a, b, c\})$*, is a colimit-dense subcategory; i.e. every object in $\mathcal{SET}$ may be written as a limit is the apex of the limit of a diagram comprising all three-element sets.*

Theorem 9.13. *Let $\mathcal{H}$ be a small, full subcategory in the complete category $\mathcal{K}$. Then the canonical functor $H : \mathcal{K} \to \mathcal{SET}^{\mathcal{H}^{op}}$ admits a left adjoint $F : \mathcal{SET}^{\mathcal{H}^{\mathrm{op}}} \to \mathcal{K}$. $I \in \mathrm{Obj}\,\mathcal{SET}^{\mathcal{H}^{\mathrm{op}}}$*

Remark 9.14. Recall that if $\mathcal{K}$ is a category and D is small, then $\Delta : \mathcal{K} \to \mathcal{K}^D$ is constant andpreserves (co)limits.

For $M \in \mathrm{Obj}\,\mathcal{K}^D$; ie. M is a diagram $M : D \to \mathcal{K}$ consider the comma-category ΔM (modelled on the pattern of $F_1 \downarrow a$). Its objects are the pairs (a, π) which are precisely cones of the diagram M.

If moreover (a, π) is a limiit of M and (b, ρ) is a cone of M, then (by the properties of limits) there eixsts precisely one $\alpha : b \to a$ such that $(\forall d \in D)\, \pi_d \circ \alpha = \rho_d$.

This may be written on as a composition of natural transformations $\pi \circ \Delta(\alpha) = \rho$.

[5] This is indeedo only because the function starts off from a monodimensional space and the map g preserves scalar multiplication which itself implicitly requries additivity in one dimension.

Theorem 9.15. *If $\mathcal{K}$ is cocomplete and $\mathcal{H} \subseteq \mathcal{K}$ is small and colimit dense, then $\mathcal{K}$ is complete.*

Proof. Let $M : D \to \mathcal{K}$ be an arbitrary diagram. Denote by $G : \mathcal{H} \to \mathcal{K}$ the (full) embedding functor. The limit of the diagram M in $\mathcal{K}$ exists iff there exists the terminal object in $\Delta \downarrow M$.

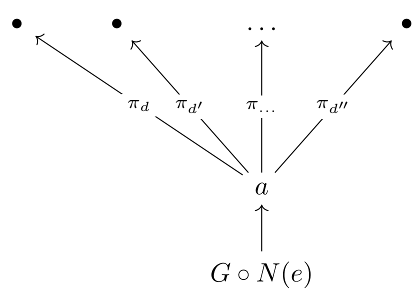
The dual theorem of the comma-category completeness (???) gives that $\Delta \downarrow M$ is cocomplete (since $\mathcal{K}$ is cocomplete and Δ admits colimits). By the dual form of the theorem which states that quasi-initial set implies initial objects (it is usually called 'initial lemma') $\Delta \downarrow M$ admits a terminal object iff there exists a quasiterminal object. This is what we need to verify.

To this end, denote $\overrightarrow{G} : \Delta \circ G \downarrow M \to \Delta \downarrow M$ is an embedding functor. Note that $\Delta \circ GM$ is a small category. Since our category is complete, $\overrightarrow{G}$ admits a colimit:

$$\operatorname{colim} \overrightarrow{G} = (\underbrace{(t, \rho)}_{\in \operatorname{Obj} \Delta \downarrow M}, \mu).$$

We shall not actually use its being a colimit, a cocone would have sufficed. Denote by $P : \Delta \downarrow M \to \mathcal{K}$ the left projection[6] We will show (t, ρ) is quasi-terminal in ΔM.

Let $(a, \pi) \in \operatorname{Obj} \Delta \downarrow M$ be arbitrary. $\mathcal{H}$ is colimit dense, whence there exists the diagram $N : E \to \mathcal{H}$ is diagram such that $(a, \nu) = \operatorname{colim}(G \circ N)$ for some suitable nu. We define the functor $A : E \to (\Delta \circ G) \downarrow M$ by $A(e) = (N(e), \pi \circ \Delta(\nu_e))$, for $e \in \operatorname{Obj} E$, $A(\lambda) = N(\lambda)$ for $\lambda \in \operatorname{Mor} E$.



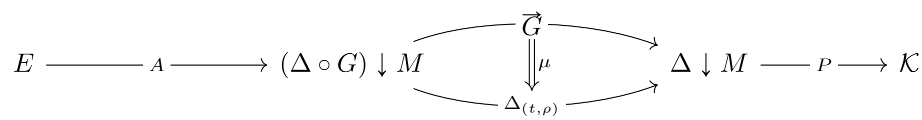
Cocone

$A(\lambda) \in \operatorname{Mor}(\delta \circ G) \downarrow M$. $\pi \circ \Delta(\nu_{e_2}) \circ \Delta GN(\lambda) = \pi \circ \Delta(\nu_{e_2} \circ GN(\lambda)) = \pi \circ \Delta(\nu_{e_1})$. Regardin the last equality, recall (a, ν) is the cocone of the diagram $G \circ N$.

A is a Functor. $A(1_e) = N(1_e) = 1_{N(e)} = 1_{A(e)}$ for some $e \in \operatorname{Obj} E$. $A(\lambda_1 \circ \lambda_2) =$

[6] $(a, \pi) \mapsto a$.

$N(\lambda_1 \circ \lambda_2) = n(\lambda_1) \circ N(\lambda_2)$.

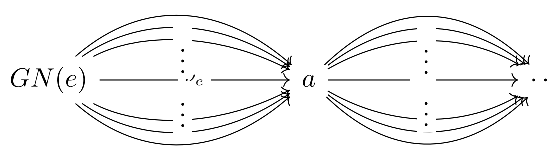


$$P\mu A \in \mathrm{Nat}(P\overrightarrow{G}A, P\Delta_{(t,\rho)}A)$$

Note $(t, P\mu A)$ is a cocone of the diagram $G \circ N$.

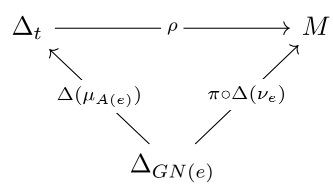
Consider $P\mu A \in \mathrm{Nat}(P \circ G \circ A, P \circ \Delta_{(t,\rho)} \circ A)$. We can see that $P \circ \overrightarrow{G} \circ A = G \circ N$. Hence $P_\mu A \in \mathrm{Nat}(G \circ N, \Delta_r)$; from the properties of colimits there exists a unique $\alpha \in \mathcal{K}(a,t)$ such that $\forall e \in \mathrm{Obj}\, E)\, \mu_{A(e)} = \alpha \circ \nu_e$. This is significant and could equivalently be written as

$$\Delta(\alpha) \circ \nu = P\mu A. \tag{2}$$



Observe $\mathrm{Mor}(\Delta \downarrow M) \ni \mu_{A(e)}$ is a colimit injection that lands in (t,ρ) whence

$$\rho \circ \Delta(_{A(e)}) = \pi \circ \Delta(\nu_e) \tag{3}$$



We shall verify that $\alpha : (a,\pi) \to (r,\rho)$ is a morphism in $\Delta \downarrow M$ such that $\rho \circ \Delta(\alpha) = \pi$; i.e. $(\forall d \in \mathrm{Obj}\, D)\, \rho_d \circ \alpha = \pi_d$. We shall verify this on the colimit injections ν_e for $e \in \mathrm{Obj}\, E$ (theorem ???). Altogether we have

$$\rho_d \circ \alpha \circ \nu_e \stackrel{2}{=} \rho_d \circ \mu_{A(e)} \stackrel{3}{=} \pi_d \circ \nu_e.$$

This holds for each $d \in \mathrm{Obj}\, D$ and each $e \in \mathrm{Obj}\, E$. We have shown $(t,\rho) \in \mathrm{Obj}\, \Delta \downarrow M$ is quasiterminal. (the categories are cocomplete and hence the existence of a quasiterminal object implies the existence of a terminal object (by analogy to theorem ??? about initial objects) and this is equivalent to the existence of a limit. hence the category is complete. QED

Note we have used the following exercise.

Exercise 9.16. Show that $\Delta\, \mathcal{K} \to \mathcal{K}$ preserves colimits for a small D.

Solution. Let $(a, \nu) = \operatorname{colim} N$ where $N : D \to \mathcal{K}$ be a diagram. We want to show that $(\delta(a), \Delta(\nu)) = \operatorname{colim}(\Delta \circ N)$. The LHS is clearly a cocone of $\Delta \circ N$. Let (G, τ) be any cocone of ΔN. Let $\tau \in (\Delta \circ N, \Delta_G)$. We are trying tofind a unique $\alpha : \Delta(a) \to G$ such that

$$(\forall e \in E)\, \alpha \circ \Delta(\nu_e) = \tau_e \in \operatorname{Nat}(\underbrace{\Delta \circ N(e)}_{\Delta_{N(a)}}, G) \qquad \text{for each } e \in \operatorname{Obj} E.$$

This is equivalent to

$$\alpha_d \circ \nu_e = (\tau_e)_d : N(e) \to G(d) \text{ for each } d \in \operatorname{Obj} D.$$

Observe that given any fixed d, $(G(d), \{(\tau_e)_d \,|\, e \in \operatorname{Obj} E\})$ is a cocone of the diagram N.

For $\lambda \in E(e_1, e_2)$, τ being a natural transformation and since $\tau_{e_2} \circ \Delta(N(\lambda)) = \tau_{e_1}$, we consequently have $(\tau_{e_2})_d \circ N(\lambda = (\tau_{e_1})_d$.

Moreover, $(G(d), (\{\tau_e)_d \,|\, e \in \operatorname{Obj} E\})$ is a cocone of the diagram N implying there exists a unique $\alpha_d : a \to G(d)$ and for each $e \in \operatorname{Obj} E$, $\alpha_d \circ \nu_e = (\tau_e)_d$. Hence we define $\alpha \stackrel{\text{def}}{=} \{\alpha_d \,|\, d \in \operatorname{Obj} D\}$. It remains to show $\alpha \in \operatorname{Nat}(\Delta_a, G)$; i.e. it remains to verify the compatibility-conditions (it is not clear if α) is a natural transformation).

For any $d \in D(d_1, d_2)$, $G(\delta) \circ \alpha_{d_1} = \alpha_{d_2} : a \to G(d_2)$; we check this on colimit injections ν_e for each $e \in \operatorname{Obj} E$:

$$G(\delta) \circ \alpha_{d_1} \circ \nu_e = G(\delta) \circ (\tau_e)_{d_1} = (\tau_e)_{d_2} = \alpha_{d_2} \circ \nu_e.$$

The second equality holds since $\tau_e \in \operatorname{Nat}(\Delta_{N(e)}, G)$.a QEF

Exercise 9.17. If $\mathcal{K}$ is a concretisible (i.e. there exists a fatihful functor $U : \mathcal{K} \to SET$) then each small category D $\mathcal{K}^D$ is likewise concretisable. This altogether implies $\mathcal{SET}^D$ is also concretisible.

10 Exercises

10.1 24^{th} November 2023

Exercise 10.1. Show that monomorphisms, epimorphisms, sections, and retractions are closed under composition.

Exercise 10.2. Consider the category $\mathcal{REL}$ whose objects are sets and whose morphisms are relations equipped with the standard operation of composition.

(1) Show that $\rho \in \mathcal{REL}(A, B)$ is a monomorphism iff there exists $B' \subseteq B$ so that $\rho \cap (A \times B')$ determined a bijection $A \to B'$.

(2) Decide when $\rho \in \mathcal{REL}(A, B)$ is an epimorphism/isomorphism.

(3) Show that all monomorphisms are sections and all epimorphisms are retractions in $\mathcal{REL}$.

Solution. HOMEWORK! QEF

Exercise 10.3. Let $(P, \leqslant)$ be a poset. Consider the category $\mathcal{P}$ whose objects are elements of the set P and morphisms correspond to ordered pairs of elements of P. This means that between any pair of objects $p, q \in \operatorname{Obj}\mathcal{P}$ there exists a morphism iff $p \leqslant q$ and this morphism is unique.

(1) Show $\mathcal{P}$ is a category.

(2) Show that every morphism in $\mathcal{P}$ is bimorphic.

(3) Show that given any morphism $f \in \mathcal{P}$ the following notions are equivalent:

   (a) f is a section.

   (b) g is a retraction.

   (c) f is an isomorphism.

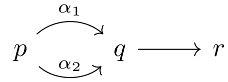
   (d) f is an identity (in some object).

(4) Let $(Q, \leqslant)$ be some other ordered set and $\mathcal{Q}$ be the corresponding category. Show that functors $\mathcal{P} \to \mathcal{Q}$ correspond to monotonic maps $P \to Q$.

   Solution.

   It easily follows from our setup that given any two $p, q \in P$, $P(p, q)$ is either empty, which occurrs if $p \nleqslant q$, or it is a singleton if $p \leqslant q$. It is easy to see the relation $\leqslant$ behaves well with respect to composition.

(2) This follows immediately from $\mathcal{P}$ being thin, which we have already established.



(3) Left out.

(4) Given a $F : \mathcal{P} \to \mathcal{Q}$ and any two morphisms $\alpha \in \mathcal{P}(a,b)$, $\beta \in \mathcal{P}(b,c)$ then $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$. But since morphisms are given by $\leqslant$, this simply means F preserves such order-relations; i.e. it is monic on P, Q.

QEF

Exercise 10.4. Let $\mathcal{POS}$ stand for a category of all ordered sets. Morphisms in this category are monotonic maps. Decide if all monomorphisms in this category are sections and if it is balanced.

Solution. Injective, monotonic maps are clearly monomorphisms; the converse statement is not so clear.

LEFT AS HOMEWORK. QEF

Exercise 10.5. Recall $\mathcal{GRP}$ is the category of groups.

(1) Show that monomorphisms in $\mathcal{GRP}$ are precisely injective homomorphisms.

(2) Show that epimorphisms in $\mathcal{GRP}$ are precisely surjective homomorphisms.

Solution.

Supppose ϕ is a non-injective monomorphism. Consider the following setup

$$\boldsymbol{K} \xrightarrow{\ j\ } \boldsymbol{G} \xrightarrow{\ \phi\ } \boldsymbol{H}$$

where $(\forall x \in K)\, j(x) = 1$, and hence $K \subseteq G$. Since ϕ is not injective, $\operatorname{Ker}\phi$ is nontrivial. Then $\phi i = \phi j \Rightarrow i = j$. A contradiction.

Let

$$\phi : \boldsymbol{G} \longrightarrow \boldsymbol{H} \underset{\psi'}{\overset{\psi}{\rightrightarrows}} \boldsymbol{P}$$

$\phi(G) = M < H$ and $\psi \neq \psi'$, $\psi \circ \phi = \psi' \circ \phi$. Suppose $[\boldsymbol{H} : \boldsymbol{M}] = 2$. Then $\boldsymbol{M} \trianglelefteq \boldsymbol{H}$. Then $\boldsymbol{P} = \mathbb{Z}_2 \simeq {}^{H}\!\diagup_{M}$ and

$$\psi' : H \to \{M\} \leqslant {}^{H}\!\diagup_{M} \qquad\qquad \psi : H \to {}^{H}\!\diagup_{M}$$

Suppose $[\boldsymbol{H} : \boldsymbol{M}] \geqslant 3$ and $\boldsymbol{P} = \operatorname{Sym}(\boldsymbol{H})$. We have at our disposal at least three distinct cosets $\boldsymbol{M}, \boldsymbol{M}u, \boldsymbol{M}v$ in ${}^{H}\!\diagup_{M}$. We define $\sigma \in P$ by

$$\begin{array}{lll} x \in M \Rightarrow & \sigma(xu) = xv & \sigma(xv) = xu \\ x \notin M \Rightarrow & \sigma(y) = y & \end{array}$$

$\psi(h)$ is a permutation given by left multiplication by h.

$\psi'(h) = b^{-1}\sigma\psi(h)\sigma$ and

$$\begin{aligned} \psi'(hh') &= \sigma^{-1}\psi(h)\psi(h')\sigma \\ &= \sigma^{-1}\psi(h)\sigma\sigma^{-1}\psi(h)\sigma \\ &= \psi(h)\psi'(h') \end{aligned}$$

Then

$$\left(\sigma^{-1}\psi(h)\sigma\right)(xu)\sigma^{-1}(\psi(h)(xv)) = \sigma^{-1}(hxv)$$

If $hxv \notin Mu \sup Mv$ then $\sigma^{-1}(hxv) = hxv$, $m \in M$ and set $h = mu^{-1}x^{-1}$. Then $hxu \in M$.

$\psi(h)(xu) = hxu$. Put

10.2 10^{th} March 2023

Exercise 10.6. Denote $\mathcal{FLD}$ the category of fields and $\mathcal{GRP}$ the category of groups. Let $Gl_n, (-)^* : \mathcal{FLD} \to \mathcal{GRP}$, defined thus:

(1) The functor Gl_n assigns the group $Gl_n(\boldsymbol{T})$ of all $n \times n$ matrices over $\boldsymbol{T}$ to the field $\boldsymbol{T}$.

(2) The functor $(-)^*$ which assigns the multiplicative group $\boldsymbol{T}^*$ of $\boldsymbol{T}$ to the field $\boldsymbol{T}$.

Solution. Observe $(-)^* = GL_1$. We firstly need to verify these indeed are functors.

Let f be a homomorphism of two fields $\boldsymbol{T}, \boldsymbol{S}$ and A, B be a $n \times n$ matrices.

$$\begin{array}{ccc} \boldsymbol{S} & \xrightarrow{\quad f \quad} & \boldsymbol{T} \\ \Big\downarrow{\scriptstyle \mathrm{GL}_n} & & \Big\downarrow{\scriptstyle \mathrm{GL}_n} \\ \mathrm{GL}_n(\boldsymbol{S}) & \xrightarrow{\mathrm{GL}_n(f)} & \mathrm{GL}_n(\boldsymbol{T}) \end{array}$$

Observe

$$\mathrm{GL}_n(f) : \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \mapsto \begin{bmatrix} f(a_{11}) & \dots & f(a_{n1}) \\ \vdots & \ddots & \vdots \\ f(a_{n1}) & \dots & f(a_{nn} \end{bmatrix}.$$

It remains to check $\mathrm{GL}_n(f)(AB) = \mathrm{GL}_n(f)(A)\mathrm{GL}_n(f)(B)$. Denote $A = \{a_{ij}\}_{i,j=1}^n$, $B = \{b_{ij}\}_{ij=1}^n$, and $C = \left\{c_{i,j=1}^n\right\}$, where $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$.

Then

$$\mathrm{GL}_n(f)(C) = \begin{bmatrix} f(c_{11}) & \dots & f(c_{1n}) \\ \vdots & \ddots & \vdots \\ f(c_{n1}) & \dots & f(c_{nn}) \end{bmatrix}$$

$$\mathrm{GL}_n(f)(A)\mathrm{GL}_n(f)(B) = \begin{bmatrix} f(a_{11}) & \dots & f(a_{n1}) \\ \vdots & \ddots & \vdots \\ f(a_{n1}) & \dots & f(a_{nn} \end{bmatrix} \cdot \begin{bmatrix} f(b_{11}) & \dots & f(b_{n1}) \\ \vdots & \ddots & \vdots \\ f(b_{n1}) & \dots & f(b_{nn} \end{bmatrix}$$

We conclude these two are the same by the arithmetic properties of the homomorphism f.

Now let us return to what we wished to prove initially. Let

$$\det_{\boldsymbol{T}} : \mathrm{GL}_n(\boldsymbol{T}) \to (\boldsymbol{T})^* \qquad\qquad A \mapsto \det(A).$$

Trivially, we obtain $\det_{\boldsymbol{T}}(AB) = \det_{\boldsymbol{T}}(A)\det_{\boldsymbol{T}}(B)$ from the fact $\det_{bsT} \in \mathrm{Mor}(\mathcal{GRP})$.

Denote

$$\det = \left\langle \det_{\boldsymbol{T}} \middle| \boldsymbol{T} \in \mathrm{Obj}(\mathcal{FLD} \right\rangle : \mathrm{GL}_n \to (\_)^*.$$

We claim this to be a natural transformation. Let $\boldsymbol{S}, \boldsymbol{T} \in \mathcal{FLD}$ and put

$$f : \boldsymbol{S} \to \boldsymbol{T}$$

$$\mathrm{GL}_n(\boldsymbol{S}) \overset{\mathrm{GL}_n(f)}{\to} \mathrm{GL}_n(\boldsymbol{T})$$

$$\boldsymbol{S}^* \overset{f^* = f^* = f\restriction S^*}{\to} \boldsymbol{T}^*$$

To see det is a natural transformation, we need to show the following diagram commutes:

$$\begin{array}{ccc} \mathrm{GL}_n(\boldsymbol{S}) & \xrightarrow{\mathrm{GL}_n(f)} & \mathrm{GL}_n(\boldsymbol{T}) \\ \Big\downarrow{\scriptstyle \det_{\boldsymbol{S}}} & & \Big\downarrow{\scriptstyle \det_{\boldsymbol{T}}} \\ \boldsymbol{S}^* & \xrightarrow{f^* = f\restriction_{\boldsymbol{S}^*}} & \boldsymbol{T}^* \end{array}$$

Both directions are readily verified

$$A \overset{\mathrm{GL}_n(f)}{\mapsto} \mathrm{GL}_n(f)(A) \overset{\det_{\boldsymbol{T}}}{\mapsto} \sum_{\pi \in \mathrm{Sym}_n} \mathrm{sgn}(\pi) f(a_{1\pi(1)}) \cdots f(a_{n\pi(n)}).$$

$$A \overset{\det_{\boldsymbol{S}}}{\mapsto} \sum_{\pi \in \mathrm{Sym}_n} \mathrm{sgn}(\pi) a_{1\pi(1)} \cdots a_{n\pi(n)} \overset{f^*}{\mapsto} f\left(\sum_{\pi \in \mathrm{Sym}_n} \mathrm{sgn} * pi) a_{1\pi(1)} \cdots a_{n\pi(n)} \right)$$

QEF

Exercise 10.7. For the set A consdier the functor $A \times - : \mathcal{SET} \to \mathcal{SET}$. Show that every map $A \to B$ determines a natural transformation $A \times - \to B \times -$.

Solution. Observe

$$\begin{aligned} A \times_{:} \mathcal{SET} &\to \mathcal{SET} \\ B &\mapsto A \times B \end{aligned}$$

and

$$\begin{aligned} (f : B \to C) \mapsto A \times f : A \times B &\to A \times C \\ \langle a, \rangle &\mapsto \langle a, f(b) \rangle . \end{aligned}$$

Let A, B be sets and $f : A \times B$. Then both $A \times -$ and $B \times -$ are functors $\mathcal{SET} \to \mathcal{SET}$.

Let $\epsilon = \langle \epsilon_c \,|\, C \in \operatorname{Obj}(\mathcal{SET}) \rangle$, $C \in \operatorname{Obj} \mathcal{SET}$ and $g : C \to D$.

$$\begin{array}{ccccc} \langle a, c \rangle & \quad & A \times C & \xrightarrow{A \times g} & A \times D \\ \big\downarrow & & \Big\downarrow {\scriptstyle \epsilon_C = f \times C} & & \Big\downarrow {\scriptstyle \epsilon_D = f \times D} \\ \langle f(a), c \rangle & & B \times C & \xrightarrow{B \times g} & B \times D \end{array}$$

The diagram commutes since

$$\begin{aligned} [(f \times D) \circ (A \times g)](a, c) &= f(\times D)(a, g(c)) = \langle f(a), g(c) \rangle \\ [(B \times g) \circ (f \times C)](a, c) &= (B \times g)(f(a), c) = \langle f(a), g(c) \rangle \end{aligned}$$

Morphisms here correspond to natural transformations on the category of functors. QEF

Exercise 10.8. Let $\mathcal{A}$ be a category and P be an ordered set (which itself may be viewed as a category). Consdier the pair of functors $F, G : \mathcal{A} \to P$.

(1) Describe when there exists a natural transformation $F \to G$.

(2) Show there exists at most one natural transformation $F \to G$.

Solution. For a natural trasformation to exist, we need for each object $a \in \operatorname{Obj} A$ a morphism $F(a) \to G(a)$ in P must exist. P is an ordered set and morphisms in P are ordered pairs. Hence $\forall a : F(a) \leqslant G(a)$. If this holds, then $F(a)$

$$\begin{array}{ccc} F(a) & \dashrightarrow & G(a) \\ \Big\downarrow {\scriptstyle F(f)} & \searrow & \Big\downarrow {\scriptstyle G(f)} \\ F(b) & \dashrightarrow & G(b) \end{array}$$

The diagram commutes iff there exists an arrow from $F(a)$ to $G(b)$. If such a morphism exists, it is uniquely determined. QEF

Exercise 10.9. Given some sets A, B, we denote B^A the set of all maps $f : A \to B$. Consider the functor $(-)^A : \mathcal{SET} \to \mathcal{SET}$. For the set X, we define the map $\epsilon_X : X^A \times A \to X$ by $\epsilon_X(f, a) = f(a)$. Show that $\epsilon = \langle \epsilon_X \mid X \in \mathrm{Obj}(\mathcal{SET}) \rangle$ is a natural transformation from the functor $(-)^A \times A$ to the identity-functor on $\mathcal{SET}$.

Solution.

$$\begin{array}{ccccccc}
\langle f, a\rangle & \longmapsto & X^A \times A & \xrightarrow{g^A \times A} & Y^A \times A & \longrightarrow & \langle g \circ f, a\rangle \\
\downarrow & & \downarrow{\scriptstyle \epsilon_x} & & \downarrow{\scriptstyle \epsilon_y} & & \downarrow \\
f(a) & \longmapsto & X & \xrightarrow{g} & Y & \longrightarrow & (g \circ f)(a)
\end{array}$$

QEF

11 Tables

11.1 Hombifunctor

Let $a, a', b, b' \in \operatorname{Obj}\mathcal{K}$ and $\alpha \in \mathcal{K}(a, a'), \beta \in \mathcal{K}(b, b')$. $\mathcal{K}(a, -)$ is *covariant* and $\mathcal{K}(-, b)$ *contravariant*.

$$F = \mathcal{K}(-,-) : \mathcal{K}^{\text{op}} \times \mathcal{K} \to \mathcal{SET}$$

(1)

$$F(a, b) = \mathcal{K}(a, b).$$

(2)

$$\begin{aligned} \mathcal{K}(a, \beta) : \mathcal{K}(a, b) &\to \mathcal{K}(a, b') \\ x &\mapsto \beta \circ x \end{aligned}$$

(3)

$$\begin{aligned} \mathcal{K}(\alpha, b) : \mathcal{K}(a', b) &\to \mathcal{K}(a, b) \\ x &\mapsto x \circ \alpha \end{aligned}$$

(4)

$$\begin{aligned} \mathcal{K}(\alpha, \beta) : \mathcal{K}(a, b) &\to \mathcal{K}(a', b') \\ \mathcal{K}(\alpha, \beta)(x) &= \beta \circ x \circ \alpha. \end{aligned}$$

11.2 Yoneda Natural Transformation

Let $\mathcal{K}$ be a category, $a \in \operatorname{Obj}\mathcal{K}$ and $F : \mathcal{K} \to \mathcal{SET}$. Set for each $b \in \operatorname{Obj}\mathcal{K}$,

$$\begin{aligned} z : F(a) &\to \operatorname{Nat}(\mathcal{K}(a, -), F) \\ x &\mapsto \tau^x \end{aligned} \qquad \begin{aligned} \tau_b^x : \mathcal{K}(a, b) &\to F(b) \\ \alpha &\mapsto [F(\alpha)](x) \end{aligned}$$