Chapter 4

Matroid Intersection Theorem — Corollaries

The Matroid Intersection Theorem provides another view on several classical results in combinatorics and graph theory. We now survey some of such results. Let us start with the one already mentioned — matchings in bipartite graphs.

Theorem 4.1. *Let G be a connected bipartite graph. The maximum size of a matching of G is equal to the minimum size of a vertex cover of G, i.e., the minimum size of s subset W of vertices of G such that each edge has at least one end-vertex in W.*

Proof. We apply the Matroid Intersection Theorem. Let V_1 and V_2 be the two vertex parts of G. The matroid $\mathcal{M}_i$, $i = 1, 2$, is defined to be the matroid with ground set formed by edges E of G and a subset $E' \subseteq E$ being independent if no two edges of E' share a vertex in V_i. Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are matroids and a maximum-size set E' independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$ is a maximum-size matching of G. By Theorem 3.4, the set E can be partitioned into sets E_1 and E_2 such that $r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2) = |E'|$. Since $r_{\mathcal{M}_1}(E_1)$ equals to the number of vertices of V_1 incident with edges of E_1, there exists a subset $U_1 \subseteq V_1$, $|U_1| = r_{\mathcal{M}_1}(E_1)$, such that each edge of E_1 has an end-vertex in U_1. Similarly, there exists a subset $U_2 \subseteq V_2$, $|U_2| = r_{\mathcal{M}_2}(E_2)$, such that each edge of E_2 has an end-vertex in U_2. Hence, the set $W = U_1 \cup U_2$ is a vertex cover of size equal to the size of E'. On the other hand, if E' is a matching of G, then the size of E' is bounded by $|W|$ for any vertex cover W of G since each edge of E' have at least one end-vertex in W. □

4.1 Analogoues of Nash-Williams Theorems

We continue with corollaries in the favor of the classical theorems of Nash-Williams on the existence of disjoint spanning trees in connected graphs and edge-covers by forests. The next theorem gives a necessary and sufficient condition for the existence of k disjoint bases in a matroid.

Theorem 4.2. *A matroid $\mathcal{M}$ with ground set E contains k disjoint bases if and only if the following holds for every subset S of its elements:*

$$|E \setminus S| \geq k(r_{\mathcal{M}}(E) - r_{\mathcal{M}}(S)) .$$

Proof. We again apply the Matroid Intersection Theorem. Let E_0 be the ground set of $\mathcal{M}$. We define two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ with ground set $E_0 = E \times \{1, \ldots, k\}$. A subset $E' \subseteq E_0$ is independent in $\mathcal{M}_1$ if each of the set $E' \cap (E \times \{i\})$ is independent in $\mathcal{M}$, i.e., $\mathcal{M}_1$ can be viewed as a union of k disjoint copies of $\mathcal{M}$. On the other hand, a subset E' is independent in $\mathcal{M}_2$ if for each element $e \in E$ there is at most one index i such that $[e, i] \in E'$, i.e., $\mathcal{M}_2$ is a transversal matroid with parts formed by the elements with the same second coordinate.

We observe that $\mathcal{M}$ has k disjoint bases if and only if there exists a set of $k \cdot r_{\mathcal{M}}(E)$ elements independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$. If $\mathcal{M}$ has k disjoint bases, say, $B_1, \ldots, B_k$, then the set $\{[e, i], e \in B_i, i = 1, \ldots, k\}$ is a set independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$. On the other hand, if there is a set E' of size $k \cdot r_{\mathcal{M}}(E)$ that is independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$, then the elements of E' with the same second coordinate form k bases of $\mathcal{M}$ that are disjoint as E' is independent in $\mathcal{M}_2$.

We are now ready to prove the statement of the theorem. If $\mathcal{M}$ contains k disjoint bases, say $B_1, \ldots, B_k$, then the following holds for every $S \subseteq E$:

$$\begin{aligned} |E \setminus S| &\geq \sum_{i=1}^{k} |B_i \cap (E \setminus S)| = \sum_{i=1}^{k} (r_{\mathcal{M}}(E) - |B_i \cap S|) \\ &\geq \sum_{i=1}^{k} (r_{\mathcal{M}}(E) - r_{\mathcal{M}}(S)) = k \cdot (r_{\mathcal{M}}(E) - r_{\mathcal{M}}(S)) . \end{aligned}$$

We next prove the converse implication.

Assume that the inequality from the statement of the theorem holds for every $S \subseteq E$. By Theorem 3.4, we have to show that for every partition of E_0 into two sets E_1 and E_2, the following holds:

$$r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2) \geq k \cdot r_{\mathcal{M}}(E) \tag{4.1}$$

If $[e, i] \in E_2$, then adding an element $[e, i']$ to E_2 does not increase the rank of E_2 in $\mathcal{M}_2$. Hence, we can assume that there exist $E' \subseteq E$ such that $E_2 = E' \times \{1, \ldots, k\}$. Consequently, $E_1 = (E \setminus E') \times \{1, \ldots, k\}$.

By the definitions of the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, $r_{\mathcal{M}_1}(E_1) = k \cdot r_{\mathcal{M}}(E \setminus E')$ and $r_{\mathcal{M}_2}(E_2) = |E'|$. Setting $S = E \setminus E'$, the assertion of the theorem implies:

$$r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2) = k \cdot r_{\mathcal{M}}(S) + |E \setminus S| \geq k \cdot r_{\mathcal{M}}(E) .$$

We conclude that the equation (4.1) holds for every partition E_1 and E_2 of E. Hence, there exists a subset of size $k \cdot r_{\mathcal{M}}(E)$ independent both in $\mathcal{M}_1$ and $\mathcal{M}_2$ and thus the matroid M has k disjoint bases by Theorem 3.4. □

A direct consequence of Theorem 4.2 is a theorem of Nash-Williams on the existence of disjoint spanning trees in graphs.

Theorem 4.3. *Let G be a connected (multi)graph. The graph G contains k edge-disjoint spanning trees if and only if for every partition $V_1, \ldots, V_\ell$ of its vertex set the number of edges between distinct parts, i.e., the number of edges uv with $u \in V_i$ and $v \in V_j$, $i \neq j$, is at least $k(\ell - 1)$.*

Proof. Assume first that G contains k disjoint spanning trees $T_1, \ldots, T_k$. Let $V_1, \ldots, V_\ell$ be any partition of the vertex set of G. Consider the graph T'_j obtained from T_j by shrinking each of the sets of V_i, $i = 1, \ldots, \ell$, to a single vertex. Since T_j is a spanning tree of G, T'_j is connected and thus it contains at least $\ell - 1$ edges. Since the trees $T_1, \ldots, T_k$ are edge-disjoint, we conclude that there are at least $k(\ell - 1)$ edges between the parts $V_1, \ldots, V_\ell$ of the partition.

In order to prove the converse implication, we apply Theorem 4.2. The matroid $\mathcal{M}$ is the graphic matroid corresponding to G and we aim to show that $\mathcal{M}$ contains k disjoint bases. Let S be a subset of edges of G and let $V_1, \ldots, V_\ell$ be the vertex sets of the components of the spanning subgraph of G with edge-set S. Note that the rank $r_{\mathcal{M}}(S)$ is equal to $n - \ell$ and the rank $r_{\mathcal{M}}(E)$ is equal to $n - 1$ where n is the order of G. Hence, $r_{\mathcal{M}}(E) - r_{\mathcal{M}}(S) = \ell - 1$. Since $E \setminus S$ contains all the edges between distinct parts $V_1, \ldots, V_\ell$ (and it can additionally contain some edges inside the parts $V_1, \ldots, V_\ell$), we infer that

$$|E \setminus S| \geq k(\ell - 1) = k\left(r_{\mathcal{M}}(E) - r_{\mathcal{M}}(S)\right) .$$

The statement now follows from Theorem 4.2. □

In the opposite direction compared to Theorem 4.2, one can ask how many independent sets are needed to cover the ground set of a given matroid. Again, it is possible to obtain a necessary and sufficient condition for the existence of a cover by k independent sets:

Theorem 4.4. *A matroid $\mathcal{M}$ with ground set E can be covered by k independent sets, i.e., there exist independent sets $E_1, \ldots, E_k$ such that $E = E_1 \cup \cdots \cup E_k$, if and only if the following holds for every subset S of its elements:*

$$|S| \leq k \cdot \left(r_{\mathcal{M}}(S)\right) .$$

Proof. If $\mathcal{M}$ can be covered by k independent sets, say, $E_1, \ldots, E_k$ then the following holds for every $S \subseteq E$:

$$|S| \leq \sum_{i=1}^{k} |E_i \cap S| \leq \sum_{i=1}^{k} r_{\mathcal{M}}(S) \leq k \cdot r_{\mathcal{M}}(S) .$$

For the proof of the converse implication, we apply the Matroid Intersection Theorem to the same matroids as in the proof of Theorem 4.2, i.e., the ground set E_0 of $\mathcal{M}_1$ and $\mathcal{M}_2$ is $E \times \{1, \ldots, k\}$, a subset E' is independent in $\mathcal{M}_1$ if each of the set $E' \cap (E \times \{i\})$ is independent in $\mathcal{M}$, and E' is independent in $\mathcal{M}_2$ if for each element $e \in E$ there is at most one index i such that $[e, i] \in E'$. Observe that the matroid $\mathcal{M}$ can be covered by k independent sets if (and only if) there exist a subset E_0 with $|E|$ elements that is independent both in $\mathcal{M}_1$ and $\mathcal{M}_2$.

Assume that the inequality from the statement of the theorem holds for every subset $S \subseteq E$. By Theorem 3.4, we have to show that for every partition of E_0 into two sets E_1 and E_2, the following holds:

$$r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2) \geq |E| \tag{4.2}$$

As in the proof of Theorem 4.2, if $[e, i] \in E_2$, then adding an element $[e, i']$ to E_2 does not increase the rank of E_2 in $\mathcal{M}_2$. Hence, we can assume that there exist $E' \subseteq E$ such that $E_2 = E' \times \{1, \ldots, k\}$. Consequently, $E_1 = (E \setminus E') \times \{1, \ldots, k\}$.

By the definitions of the matroids $\mathcal{M}_1$ and $\mathcal{M}_2$, $r_{\mathcal{M}_1}(E_1) = k \cdot r_{\mathcal{M}}(E \setminus E')$ and $r_{\mathcal{M}_2}(E_2) = |E'|$. By the assumption of the theorem, $|E \setminus E'| \leq k \cdot r_{\mathcal{M}}(E \setminus E')$, i.e., $|E'| \geq |E| - k \cdot r_{\mathcal{M}}(E \setminus E')$. Plugging this inequality in the sum $r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2)$, yields the following:

$$r_{\mathcal{M}_1}(E_1) + r_{\mathcal{M}_2}(E_2) = kr_{\mathcal{M}}(E \setminus E') + |E'| \geq |E|; .$$

We conclude that the inequality (4.2) holds for every partition E_1 and E_2 of E and thus the matroid $\mathcal{M}$ can be covered by k independent sets. □

A graph-theoretic analogue of Theorem 4.4 is another theorem of Nash-Williams. We state the theorem and omit its proof as it follows the lines of the proof of Theorem 4.3.

Theorem 4.5. *Let G be a (multi)graph. The edges of the graph G can be covered by k forests if and only if for every subset W of the vertices of G the subgraph $G[W]$ induced by W contains at most $k(|W| - 1)$ edges.*

Another corollary of the Matroid Intersection Theorem is a necessary and sufficient condition on the existence of a rainbow spanning tree in a graph.

Theorem 4.6. *Let G be a graph with colored edges. G contains a rainbow spanning tree, i.e., a spanning tree with edges of mutually distinct colors, if and only if $G \setminus E'$ has at most t components for every set E' of edges with at most $t - 1$ colors.*

Proof. We will apply Theorem 3.4. The matroid $\mathcal{M}_1$ is the graphic matroid of G. The matroid $\mathcal{M}_2$ is the partition matroid where a set of edges E' is independent if the edges of E' have mutually distinct colors, i.e., $\mathcal{M}_2$ is a transversal matroid with respect to the color classes of edges. The existence of a rainbow spanning tree is now equivalent to the existence of a set of $n-1$ edges independent both in $\mathcal{M}_1$ and $\mathcal{M}_2$ where n is the order of G. The latter is equivalent by Theorem 3.4 to the assertion that the following equality holds:

$$\min_{E'\subseteq E} r_{\mathcal{M}_1}(E') + r_{\mathcal{M}_2}(E\setminus E') = n-1 \tag{4.3}$$

where E is the set of all edges of G.

Assume that the number of components of $G\setminus E'$ does not exceed t for every set E' of edges with at most $t-1$ colors. Consider a subset $E'\subseteq E$, let $E''=E\setminus E'$ and let $t-1$ be the number of the colors of the edges of E''. Our assumption implies that the number of components of $G\setminus E''$ is at most t and thus $r_{\mathcal{M}_1}(E\setminus E'') = r_{\mathcal{M}_1}(E') \ge n-t$. On the other hand, the definition of $\mathcal{M}_2$ implies that $r_{\mathcal{M}_2}(E\setminus E') = r_{\mathcal{M}_2}(E'') = t-1$. This yields that the minimum in (4.3) is at least $n-1$. As $r_{\mathcal{M}_1}(E) = n-1$ and $r_{\mathcal{M}_2}(\emptyset)=0$, the equality (4.3) holds and a rainbow spanning tree exists by Theorem 3.4.

In the reverse direction, if T is a rainbow spanning tree and E' is a set of edges with $t-1$ colors, then $G\setminus E'$ has at most t components as the components of $G\setminus E'$ must be interconnected by edges of $E'\cap T$; since T is rainbow, there are at most $t-1$ edges contained in $E'\cap T$ which can interconnect the components. This finishes the proof of the theorem. □

4.2 Maximum average degree of graphs

We now present another (more algorithmic) application of Theorem 4.4. An important graph-theoretic parameter is the maximum average degree of a subgraph of G which is equal to the following quantity:

$$\mathrm{mad}(G) = \max_{G'\subseteq G} \frac{2||G'||}{|G'|}$$

where $|G'|$ is the order and $||G'||$ is the size of G'. Observe any graph G with $\mathrm{mad}(G)\le d$ is d-degenerate, i.e., every subgraph of G has a vertex of degree at most d, and any d-degeneated graph G has $\mathrm{mad}(G)\le 2d$. Somewhat surprisingly, the maximum average degree of a given graph G can be computed in polynomial time. Before we do so, we need an auxiliary lemma.

Lemma 4.7. *Let G be a graph with edge set E and let $\mathcal{I}$ be the family of subsets E' of E such that the subgraph of G formed by the edges in E' contains at most one cycle. The pair $(E,\mathcal{I})$ forms a matroid. Moreover, the rank of E' is equal to $r_{\mathcal{M}(G)}(E')$ if E' is acyclic, and to $r_{\mathcal{M}(G)}(E')+1$, otherwise.*

Proof. In order to show that $(E,\mathcal{I})$ is a matroid, it is enough to establish that the function $r(E')$, $E'\subseteq E$, defined to be $r_{\mathcal{M}(G)}(E')$ if E' is acyclic, and to

$r_{\mathcal{M}(G)}(E')+1$, otherwise, satisfies the properties (R1), (R2) and (R3) given in Lemma 1.9. Since (R1) and (R2) are obvious, we focus on (R3). Let E' and E'' be two subsets of E. We have to show that

$$r(E'\cap E'')+r(E'\cup E'')\le r(E')+r(E'') .$$

If neither E' nor E'' is acyclic, then $r(E')+r(E'') = r_{\mathcal{M}}(E')+r_{\mathcal{M}}(E'')+2$. Since $r(E'\cap E'')+r(E'\cup E'')\le r_{\mathcal{M}}(E'\cap E'')+r_{\mathcal{M}}(E'\cup E'')+2$, the inequality holds. If one of E' and E'' is acyclic and the other is not, then $r(E')+r(E'') = r_{\mathcal{M}}(E')+r_{\mathcal{M}}(E'')+1$. Since $E'\cap E''$ is a subset of both E' and E'', $E'\cap E''$ is acyclic and thus $r(E'\cap E'') = r_{\mathcal{M}}(E'\cap E'')$. Consequently, $r(E'\cap E'')+r(E'\cup E'')\le r_{\mathcal{M}}(E'\cap E'')+r_{\mathcal{M}}(E'\cup E'')+1$ and the inequality holds.

We now assume that both E' and E'' are acyclic. Hence, $r(E')+r(E'') = r_{\mathcal{M}}(E')+r_{\mathcal{M}}(E'') = |E'|+|E''|$ and $r(E'\cap E'') = r_{\mathcal{M}}(E'\cap E'') = |E'\cap E''|$. If $r(E'\cup E'') = r_{\mathcal{M}}(E'\cup E'')$, the inequality holds. Otherwise, $E'\cup E''$ must contain a cycle which implies that $r(E'\cup E'') = r_{\mathcal{M}}(E'\cup E'')+1\le |E'\cup E''|-1+1 = |E'\cup E''|$. Since $|E'|+|E''| = |E'\cap E''|+|E'\cup E''|$, the inequality follows. □

Denote the matroid introduced in Lemma 4.7 by $\mathcal{M}^+(G)$. The easiest way to show that the maximum average degree of a graph can be computed in polynomial time is to employ Lemma 4.7 and the theory on matroid polytopes. Here, we present a less direct argument which however uses the theory we have developed so far.

Theorem 4.8. *There is a polynomial-time algorithm for computing the maximum average degree of a graph.*

Proof. Let n and m be the order and the size of an input graph G. By the definition, the maximum average degree is equal to $2k/\ell$ for some $0\le k\le m$ and $1\le \ell\le n$. As the number of such fractions $2k/\ell$ is at most $O(nm)$, it is enough to present a polynomial-time algorithm for testing whether $\mathrm{mad}(G)\le 2k/\ell$ for every such fraction $2k/\ell$.

Let H be a graph obtained from G by replacing each edge by ℓ parallel edges. Theorem 4.4 yields that $\mathcal{M}^+(H)$ can be covered by k independent sets if and only if $|F|\le kr_{\mathcal{M}^+(H)}(F)$ for every subset F of edges of H. Observe $r_{\mathcal{M}^+(H)}(F)$ does not exceed the number of vertices of the subgraph of H spanned by F by Lemma 4.7. In particular, if $\mathrm{mad}(G)\le 2k/\ell$ which is equivalent to $\mathrm{mad}(H)\le 2k$, then $|F|\le k|V(H[F])| = kr_{\mathcal{M}^+(H)}(F)$ for every subset F containing a cycle. If F is acyclic then $|F| = r_{\mathcal{M}^+(H)}(F)$ and the inequality $|F|\le kr_{\mathcal{M}^+(H)}(F)$ holds. Thus the ground set of $\mathcal{M}^+(H)$ can be covered by k sets independent in $\mathcal{M}^+(H)$. In the other direction, if $\mathcal{M}^+(H)$ can be covered by k sets independent in $\mathcal{M}^+(H)$, then every n'-vertex subgraph of H contains at most n' edges of each of these k sets by Lemma 4.7. In particular, its average degree does not exceed $2k$ and thus $\mathrm{mad}(H)\le 2k$ which is equivalent to $\mathrm{mad}(G)\le 2k/\ell$. We conclude that

$\mathrm{mad}(G) \leq 2k/\ell$ if and only if $\mathcal{M}^+(H)$ can be covered by k independent sets. Since this is a particular case of the Matroid Intersection Theorem as outlined in the proof of Theorem 4.4, there is a polynomial-time algorithm to decide whether the inequality $||G'|| \leq k|G'|/\ell$ holds for every subgraph G' of G. Since the number of choice of k and ℓ is polynomial in $O(nm)$, we conclude that the average degree of a graph can be computed in polynomial time. □

4.3 Common transversals

As our final application of the Matroid Intersection Theorem, we establish a necessary and sufficient condition for the existence of a common transversal of two set systems.

Theorem 4.9. *Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ be two set systems with the same ground set. The two set systems have a common transversal, i.e., a set $Z = \{x_1, \ldots, x_m\} = \{y_1, \ldots, y_m\}$ of m elements such that $x_i \in X_i$ and $y_i \in Y_i$, $i = 1, \ldots, m$, if and only if the following holds for every $I \subseteq \{1, \ldots, m\}$ and every $J \subseteq \{1, \ldots, m\}$:*

$$\left| \left(\bigcup_{i \in I} X_i \right) \cap \left(\bigcup_{j \in J} Y_j \right) \right| \geq |I| + |J| - m \ .$$

Proof. The condition in the statement of the theorem is clearly necessary: if a set Z of m elements that is transversal of both the set systems exist, then the intersection $Z \cap \bigcup_{i \in I} X_i$ contains at least $|I|$ elements of Z and the intersection $Z \cap \bigcup_{j \in J} Y_j$ contains at least $|J|$ elements of Z. If $|I| + |J| > m$, then the two intersections must have at least $|I| + |J| - m$ elements in common. Hence, the condition is necessary for the existence of a common transversal of the set systems.

To establish the sufficiency, we will apply the Matroid Intersection Theorem. If the sets $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ were disjoint, they would form partitions of the ground set and it would be easy to apply the Matroid Intersection Theorem to the corresponding transversal matroids. However, the sets need not be disjoint and we will have to define a matroid in a less direct way.

Let us start with the definition of a matroid $\mathcal{M}_X$ which corresponds to the set system $X_1, \ldots, X_m$. The rank $r_{\mathcal{M}_X}(Z)$ of a subset Z of the ground set is equal to the maximum cardinality of an index set $I \subseteq \{1, \ldots, m\}$ such that Z contains a transversal of the set system X_i, $i \in I$. We have to verify that the just defined rank function satisfies the properties (R1), (R2) and (R3) given in Lemma 1.9. (R1) and (R2) are straightforward to verify. We now focus on establishing that the function $r_{\mathcal{M}_X}$ is submodular, i.e., we have to show that any two sets Z_1 and Z_2 satisfy the following inequality:

$$r_{\mathcal{M}_X}(Z_1 \cap Z_2) + r_{\mathcal{M}_X}(Z_1 \cup Z_2) \leq r_{\mathcal{M}_X}(Z_1) + r_{\mathcal{M}_X}(Z_2) \tag{4.4}$$

Let $I^\cap$ be a maximum index set such that $Z_1 \cap Z_2$ contains a transversal of X_i, $i \in I^\cap$, and let $z_i^\cap \in Z_1 \cap Z_2$ be one of the transversals, i.e., $z_i^\cap \in X_i$. Similarly, the set $I^\cup$ and the elements $z_i^\cup$ are defined with respect to $Z_1 \cup Z_2$. Finally, let $Z^\cap = \{z_i^\cap, i \in I^\cap\}$ and $Z^\cup = \{z_i^\cup, i \in I^\cup\}$. Observe that we can assume without loss of generality that $I^\cap \subseteq I^\cup$ (this is non-trivial and requires revisiting Hall's theorem!).

Define an auxiliary bipartite graph H with parts $Z^\cap$ and $Z^\cup$ that has two types of edges:

- for every $i \in I^\cap \cap I^\cup = I^\cap$, the elements $z_i^\cap \in Z^\cap$ and $z_i^\cup \in Z^\cup$ are joined by an edge, and
- for $z \in Z^\cap \cap Z^\cup$, the two copies of z are joined by an edge.

Clearly, H consists of isolated vertices, paths and cycles. Moreover, for every component P of H that is a path, at most one of its ends lies in $Z^\cap$: if P contains a single edge, there is nothing to prove. If P has more edges, the two types of edges alternate on P. If P ends at a vertex of the part corresponding to $Z^\cap$, the edge incident with this vertex is of the first type (because each vertex of $Z^\cap$ contains one edge of this type). Hence, both ends of P cannot lie in the part of H corresponding to $Z^\cap$ as claimed.

We now color the vertices of H with two colors in such a way that a vertex corresponding to an element z can get a color i only if $z \in Z_i$ and the coloring is a proper coloring of Z. Let us examine whether each component C of H can be colored in this way. If C is a cycle, then half of its edges are of the second type and thus all vertices on C correspond to elements in $Z_1 \cap Z_2$; in particular, any proper coloring of the vertices of C with colors 1 and 2 satisfies our requirements. If C is a path, all the vertices of C with a possible exception for one end-vertex of C can get both colors 1 and 2. We first color the exceptional end-vertex of C (if it does not exist, any end-vertex of C) and then extend the coloring to C.

Let I_1 be the set of indices such that $z_i^\cap$ or $z_i^\cup$ (if they exist) are colored with the color 1 and I_2 the set of indices such that one of these two vertices is colored with the color 2. Because of the edges in H of the first type, the sum $|I_1|+|I_2|$ is equal to $|I^\cup|+|I^\cap|$. On the other hand, the elements corresponding to the vertices of H colored with the first color are mutually distinct because of the second type of edges in H and thus they form a transversal of the set system X_i, $i \in I_1$. Similarly, the elements corresponding to the vertices colored with the color 2 form a transversal of the set system X_i, $i \in I_2$. This implies that $r_{\mathcal{M}_X}(Z_1) \geq |I_1|$ and $r_{\mathcal{M}_X}(Z_2) \geq |I_2|$ which yields (4.4). This finishes the proof that the function $\mathcal{M}_X$ is a rank function and thus the matroid $\mathcal{M}_X$ is well-defined. In an analogous way, we define a matroid $\mathcal{M}_Y$ for the set system $Y_1, \ldots, Y_m$.

We can now apply the Matroid Intersection Theorem for the matroids $\mathcal{M}_X$ and $\mathcal{M}_Y$. To this end, we have to show that for every subset Z of the ground set that the sum $r_{\mathcal{M}_X}(Z) + r_{\mathcal{M}_Y}\left(\overline{Z}\right)$ is at least m where $\overline{Z}$ is the complement of Z. Assume that there is a subset Z such that

$$r_{\mathcal{M}_X}(Z) + r_{\mathcal{M}_Y}\left(\overline{Z}\right) < m \tag{4.5}$$

If $r_{\mathcal{M}_X}(Z) = m - d_X$ and $r_{\mathcal{M}_Y}\left(\overline{Z}\right) = m - d_Y$, then Hall's theorem yields the existence of subsets $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, m\}$ such that

$$\left|\bigcup_{i\in I}(X_i \cap Z)\right| = |I| - d_X \qquad \text{and} \qquad \left|\bigcup_{j\in J}(Y_j \cap \overline{Z})\right| = |J| - d_Y \ .$$

This consequently implies that

$$\left|Z \cap \left(\bigcup_{i\in I} X_i\right)\right| = |I| - d_X \qquad \text{and} \qquad \left|\overline{Z} \cap \left(\bigcup_{j\in J} Y_j\right)\right| = |J| - d_Y \ .$$

Finally, we obtain that

$$\left|Z \cap \left(\bigcup_{i\in I} X_i\right) \cap \left(\bigcup_{j\in J} Y_j\right)\right| \leq \left|Z \cap \left(\bigcup_{i\in I} X_i\right)\right| = |I| - d_X \tag{4.6}$$

and that

$$\left|\overline{Z} \cap \left(\bigcup_{i\in I} X_i\right) \cap \left(\bigcup_{j\in J} Y_j\right)\right| \leq \left|\overline{Z} \cap \left(\bigcup_{j\in J} Y_j\right)\right| = |J| - d_Y \ . \tag{4.7}$$

Since Z and $\overline{Z}$ form a partition of the ground set, the inequalities (4.6) and (4.7) combine to the following estimate (last inequality follows from (4.5) and the choice of d_X and d_y):

$$\left|\left(\bigcup_{i\in I} X_i\right) \cap \left(\bigcup_{j\in J} Y_j\right)\right| \leq |I| + |J| - d_X - d_Y \leq |I| + |J| - m - 1 \tag{4.8}$$

By the assumption of the theorem, the just obtained inequality cannot hold and thus there is no set Z satisfying (4.5). Hence, the Matroid Intersection Theorem yields the existence of the common transversal of the set systems $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$. □