Name and surname: ______________________

Problem	1	2	3	4	**Total points**
Points	8	7	8	7	30
Points earned					

[8] 1. **Operator and its properties**

On the space $L^2((0,1))$ over $\mathbb{R}$, consider the operator L defined through

$$(Lf)(x) := \int_0^1 k(x,s) f(s)\,\mathrm{d}s,$$

where $k \in \mathcal{C}([0,1]\times[0,1])$ is a continuous real map satisfying

$$k(x,y) = k(y,x) \qquad \text{for all } x, y \in [0,1]\,. \tag{1}$$

(a) Prove that $L : L^2((0,1)) \to L^2((0,1))$ is a compact selfadjoint linear bounded operator. If you apply any theorem, write down its formulation and verify its assumptions.

(b) As a special case verify that the assumption (1) is satisfied by the integral kernel k of the form

$$k(x,y) := \begin{cases} (1-y)x & \text{if } 0 \le x \le y\,, \\ (1-x)y & \text{if } y \le x \le 1. \end{cases} \tag{2}$$

(c) Provided that f is continuous on $(0,1)$ show that function $u(x) = (Lf)(x)$ with k given by (2) solves the problem

$$u''(x) + f(x) = 0\,, \qquad u(0) = u(1) = 0\,.$$

(d) Referring to the course on classical partial differential equations, do you know what is the name of the kernel k?

Solution:

(a) L is clearly linear. To establish the boundedness we first estimate using Hölder's inequality:

$$\left|\int_0^1 k(x,s)f(s)\,\mathrm{d}s\right|^2 \le \int_0^1 |k(x,s)|^2\,\mathrm{d}s \int_0^1 |f(s)|^2\,\mathrm{d}s = \|f\|_2^2 \int_0^1 |k(x,s)|^2\,\mathrm{d}s.$$

This implies that

$$\|Lf\|_2^2 = \int_0^1 \left|\int_0^1 k(x,s)f(s)\,\mathrm{d}s\right|^2 \mathrm{d}x \le \|f\|_2^2 \int_{(0,1)^2} |k(x,s)|^2\,\mathrm{d}s\,\mathrm{d}x.$$

The integral of k over the unit square is finite as k is continuous on $[0,1]^2$, so L is bounded. (In fact, we can see that $k \in L^2\big((0,1)^2\big)$ would be enough for boundedness of L.)

In the sequel $(\cdot,\cdot)$ denotes the inner product on $L^2(0,1)$, i.e., $(f,g) = \int_0^1 f(x)\,g(x)\,\mathrm{d}x$. For arbitrary $f,\ g \in L^2(0,1)$ we have

$$\begin{aligned}(Lf,g) = \int_0^1 (Lf)(x)\,g(x)\,\mathrm{d}x &= \int_0^1 \Big(\int_0^1 k(x,s)\,f(s)\,\mathrm{d}s\Big)\,g(x)\,\mathrm{d}x \\ &= \int_{(0,1)^2} k(x,s)\,f(s)\,g(x)\,\mathrm{d}s\,\mathrm{d}x = \int_{(0,1)^2} k(s,x)\,f(s)\,g(x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^1 f(s)\,\Big(\int_0^1 k(s,x)\,g(x)\,\mathrm{d}x\Big)\,\mathrm{d}s = \int_0^1 f(s)\,(Lg)(s)\,\mathrm{d}s = (f,Lg),\end{aligned}$$

where we applied Fubini's theorem twice and on the middle line we used the symmetry $k(x,s) = k(s,x)$. Thus L is self-adjoint.

To show that L is compact, it suffices, given an arbitrary bounded sequence $\{f_n\}_{n=1}^\infty \subset L^2(0,1)$, to show that $\{Lf_n\}_{n=1}^\infty$ is precompact in $L^2(0,1)$. Precompactness in Lebesgue spaces is characterized by the Fréchet–Kolmogorov–M. Riesz criterion. $\{Lf_n\}$ is bounded by virtue of continuity of L, equitightness is moot on a bounded domain, so it remains to establish equicontinuity $\|Lf_n(\cdot - h) - Lf_n\|^2_{L^2(\mathbb{R})} \to 0$ as $h \to 0$ uniformly in n, where Lf_n is extended by zero outside of $(0,1)$. But this follows easily:

$$\begin{aligned}\|Lf(\cdot - h) - Lf\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_0^1 k(x-h,s) f(s)\, \mathrm{d}s - \int_0^1 k(x,s) f(s)\, \mathrm{d}s \right|^2 \\ &= \int_{\mathbb{R}} \left| \int_0^1 \big(k(x-h,s) - k(x,s)\big) f(s)\, \mathrm{d}s \right|^2 \le \|k(\cdot - h, \cdot) - k\|^2_{L^2(\mathbb{R}\times(0,1))} \|f\|_2^2,\end{aligned}$$

where k has been extended to $\mathbb{R} \times (0,1)$ be zero and the inequality follows from Hölder's inequality and Fubini's theorem. As the right-hand side is uniform in $\|f\|_2$, the equicontinuity follows from $\lim_{h\to 0} \|k(\cdot - h, \cdot) - k\|_{L^2(\mathbb{R}\times(0,1))} = 0$. This is indeed true by, say, Lebesgue's dominated convergence theorem and continuity of k, or it follows from the uniform continuity of k on $[0,1]^2$.

(b) Computing explicitly $k(y,x)$ using (2) gives

$$k(y,x) = \begin{cases} (1-x)y & \text{if } 0 \le y \le x \le 1, \\ (1-y)x & \text{if } 0 \le x \le y \le 1, \end{cases}$$

which is clearly equal to $k(x,y)$.

(c) Inserting (2) in the definition of L we obtain

$$u(x) = (1-x) \int_0^x s\, f(s)\, \mathrm{d}s + x \int_x^1 (1-s)\, f(s)\, \mathrm{d}s.$$

Evaluating both terms at the boundary shows that $u(0) = u(1) = 0$. For the first derivative we have

$$u'(x) = -\int_0^x s\, f(s)\, \mathrm{d}s + (1-x)\, x\, f(x) + \int_x^1 (1-s)\, f(s)\, \mathrm{d}s - x\,(1-x)\, f(x) = -\int_0^x s\, f(s)\, \mathrm{d}s + \int_x^1 (1-s)\, f(s)\, \mathrm{d}s$$

and in turn for the second derivative

$$u''(x) = -x\, f(x) - (1-x)\, f(x) = -f(x).$$

As f, and in turn all the integrands in the expressions for $u(x)$ and $u'(x)$, are now continuous, the last equality holds everywhere in $(0,1)$.

(d) *Green's function.*

[7] 2. **Spectrum**

Let $X := \big(C([0,1]), \|\cdot\|_\infty\big)$. Consider $L\colon X \to X$ defined through

$$(Lf)(x) := f(0) 2(1-x)(\tfrac{1}{2} - x) + f(\tfrac{1}{2}) 4(1-x)x + f(1) 2x(x - \tfrac{1}{2}).$$

(a) Is the operator L (i) linear, (ii) bounded, (iii) compact?

(b) Give the definition of the spectrum of L (denoted $\sigma(L)$) and the point spectrum of L (denoted $\sigma_p(L)$), and determine them for the above operator.

(c) For each eigenvalue, if any, determine the corresponding eigenspace.

Solution:

(a) The operator is clearly linear. To show boundedness, we estimate

$$\begin{aligned}\|Lf\|_\infty &= \max_{x\in[0,1]}\left|f(0)2(1-x)(\tfrac{1}{2}-x)+f(\tfrac{1}{2})4(1-x)x+f(1)2x(x-\tfrac{1}{2})\right|\\ &\le \max_{x\in[0,1]}\left(|f(0)2(1-x)(\tfrac{1}{2}-x)|+|f(\tfrac{1}{2})4(1-x)x|+|f(1)2x(x-\tfrac{1}{2})|\right)\\ &\le \max\{|f(0)|,|f(\tfrac{1}{2})|,|f(1)|\}\max_{x\in[0,1]}\left(|2(1-x)(\tfrac{1}{2}-x)|+|4(1-x)x|+|2x(x-\tfrac{1}{2})|\right)\\ &\le \|f\|_\infty \max_{x\in[0,1]}\left(|2(1-x)(\tfrac{1}{2}-x)|+|4(1-x)x|+|2x(x-\tfrac{1}{2})|\right),\end{aligned}$$

which shows that the operator is bounded due to $\max_{x\in[0,1]}\left(|2(1-x)(\frac{1}{2}-x)|+|4(1-x)x|+|2x(x-\frac{1}{2})|\right)<\infty$. Now observe that the range of L is a subspace of $\mathcal{P}_2$, the space of polynomials of degree at most 2, and hence it is at most three-dimensional. Compactness of L then follows from the fact that a bounded linear operator with finite-dimensional range is compact.

(b) For a bounded linear operator T on a Banach space, its spectrum $\sigma(T)$ is defined as the set of all complex numbers such that $T-\lambda I$ is not invertible and its point spectrum $\sigma_{\mathrm{p}}(T)$ is defined as the set of all complex numbers such that $T-\lambda I$ is not injective.

As the operator is compact, we have $\sigma(L)=\{0\}\cup\sigma_{\mathrm{p}}(L)$ with $\sigma_{\mathrm{p}}(L)$ at most countable, with only limit point being possibly zero. Moreover, the elements of $\sigma_{\mathrm{p}}(L)\setminus\{0\}$ are eigenvalues of finite algebraic multiplicity. Nevertheless, for the solution of this exam problem we only need the relation $\sigma(L)=\{0\}\cup\sigma_{\mathrm{p}}(L)$. It remains to find the eigenvales $\sigma_{\mathrm{p}}(L)$.

Let us first investigate whether 0 belongs to $\sigma_{\mathrm{p}}(L)$. It would mean that there is a non-trivial solution of the equation $Lf=0$. This is indeed the case: $\ker L=\{f\in C([0,1]),\ f(0)=f(\frac{1}{2})=f(1)=0\}$, an infinite-dimensional space.

As of non-zero eigenvalues, the eigenequation $Lf=\lambda f$, $\lambda\ne 0$, reads

$$f(0)2(1-x)(\tfrac{1}{2}-x)+f(\tfrac{1}{2})4(1-x)x+f(1)2x(x-\tfrac{1}{2})=\lambda f(x)\qquad \text{for all } x\in[0,1]. \tag{3}$$

This, in particular, implies that f needs to be a polynomial of degree at most two. Evaluating (3) at 0, $\frac{1}{2}$, and 1 we obtain equations $f(0)=\lambda f(0)$, $f(\frac{1}{2})=\lambda f(1)$, and $f(1)=\lambda f(1)$. This implies that the equation can only be satisfied with $\lambda=1$. With f an arbitrary polynomial of degree at most 2 and $\lambda=1$, the equation (3) is satisfied at three distinct points, namely 0, $\frac{1}{2}$, and 1, and both sides of the equation are polynomials of degree at most 2. Hence, the equation is satisfied everywhere.

Altogether $\sigma(L)=\sigma_{\mathrm{p}}(L)=\{0,1\}$.

(c) We have shown above that $\ker L=\{f\in C([0,1]),\ f(0)=f(\frac{1}{2})=f(1)=0\}$. We have also shown that (3) is satisfied for arbitrary $f\in\mathcal{P}_2$, the space of polynomials of degree at most 2, and $\lambda=1$. Hence $\ker(L-I)=\mathcal{P}_2$. It was not possible to fulfill (3) in any other way, so these are all eigenspaces.

The solution is now complete but we would like to mention it was worth noticing that L is Lagrange interpolation at equidistant nodes $x_0=0$, $x_1=\frac{1}{2}$, and $x_2=1$. Denote $\ell_0(x)=2(1-x)(\frac{1}{2}-x)$, $\ell_1(x)=4(1-x)x$, and $\ell_2(x)=2x(x-\frac{1}{2})$. Then $\{\ell_0,\ell_1,\ell_2\}$ is the Lagrange basis of $\mathcal{P}_2$, i.e., $\operatorname{span}\{\ell_0,\ell_1,\ell_2\}=\mathcal{P}_2$ and $\ell_i(x_j)=\delta_{ij}$. The operator can be expressed as $Lf(x)=\sum_{i=0,1,2}f(x_i)\ell_i(x)$. We easily observe that the range of L equals $\mathcal{P}_2$ and that the operator is idempotent (a projection), i.e., $L^2f=Lf$ for all $f\in X$. (Recall that the Lagrange interpolant with 3 nodes of a function from $\mathcal{P}_2$ is the same function.) This yields immediatelly that $\ker(L-I)=\mathcal{P}_2$. This way, the eigenvalue problem (3) has been solved much easier.

Note that the inequality, which we obtained above, $\|L\|\le\max_{x\in[0,1]}\sum_{i=0,1,2}|\ell_i(x)|$ is actually an equality and $\|L\|$ is called the Lebesgue constant (for the nodes x_0, x_1, x_2). It requires tedious but simple calculation to show that $\|L\|=\frac{5}{4}$ for the given nodes x_0, x_1, x_2.

[8] 3. **Duals**

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces over the scalar field $\mathbb{K}$.

(a) Define $\mathcal{L}(X,Y)$, $\mathcal{L}(X)$ and X^*. Prove that X^* is a Banach space.

(b) Show that the mapping $\phi\to\int_0^1 u(x)\phi(x)\,\mathrm{d}x$ is, for a given $u\in L^{p'}((0,1))$, $p'=p/(p-1)$, element of $[L^p((0,1))]^*$?

(c) If $1<p<\infty$, can you give an example of an element of $[L^p((0,1))]^*$ that is of a different form than that given in the above point?

(d) Prove the following.

Proposition 1. *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be normed spaces such that $X\subset Y$ and there exists $C>0$ such that $\|x\|_Y\le C\|x\|_X$ for every $x\in X$. Then it holds that $Y^*\subset X^*$ and $\|F\|_{X^*}\le C\|F\|_{Y^*}$ for every $F\in Y^*$.*

Next, by means of Hölder's inequality, show that $L^4((0,1)) \subset L^2((0,1))$. What kind of relation between the spaces $L^2((0,1))$ and $L^{4/3}((0,1))$ can one conclude using Proposition 1 and the characterization of duals of Lebesgue spaces?

(e) Give a dual description of $\|x\|_X$ and prove that this description holds.

Solution:

(a) For normed spaces X and Y over $\mathbb{K}$, we denote by $\mathcal{L}(X,Y)$ the set of all continuous linear mappings from X to Y. Then we set $\mathcal{L}(X) = \mathcal{L}(X,X)$ and $X^* = \mathcal{L}(X,\mathbb{K})$.

It is straightforward to verify that the set X^* is a vector space with addition and multiplication by scalars defined naturally pointwise, i.e., for f, $g \in X^*$, and $\alpha \in \mathbb{K}$, we define a map $f + \alpha g\colon X \to \mathbb{K}$ by

$$(f+\alpha g)(x) = f(x) + \alpha g(x) \qquad \text{for all } x \in \mathbb{K}.$$

Clearly $f + \alpha g$ is linear and continuous, which ensures that $f+\alpha g$ is an element of X^*, and thus X^* is a vector space. For every $f \in X^*$ we define its norm $\|f\|_{X^*} := \sup_{x\in X,\, \|x\|\le 1} \|f(x)\|_X$. It can be verified that this renders X^* a normed space. It remains to verify its completeness.

Consider a Cauchy sequence $\{f_n\}_{n=1}^\infty \subset X^*$. For an arbitrary $x \in X$, the number sequence $\{f_n(x)\}_{x=1}^\infty$ is Cauchy too, which follows from the estimate $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_{X^*}\|x\|_X$. As long as $\mathbb{K}$ is complete, the sequence $\{f_n(x)\}$ converges and we denote its limit as $f(x)$. By the arithmetic of limits, $x \mapsto f(x)$ is a linear map and as $\{f_n(x)\}$ is Cauchy uniformly on $\{x \in X,\ \|x\|_X \le 1\}$ and f_n is continuous, it is obtained by the standard 3ϵ argument that f is continuous, i.e., $f \in X^*$. As the limit $f_n(x) \to f(x)$ is uniform on $\{x \in X,\ \|x\|_X \le 1\}$, it is easily established that $\|f_n - f\|_{X^*} \to 0$. (Note that this proof actually needs no modification in order to show that, for Y Banach, $\mathcal{L}(X,Y)$ is Banach. Instead of $(\mathbb{K}, |\cdot|)$ take $(Y, \|\cdot\|_Y)$.)

(b) Denote the mapping $\phi \mapsto \int_0^1 u\phi$ by T_u. It is clearly linear. Let us show that it is defined on all $L^p(0,1)$ and is bounded:

$$|T_u\phi| = \left|\int_0^1 u\,\phi\right| \le \int_0^1 |u\,\phi| \le \|u\|_{L^{p'}(0,1)}\|\phi\|_{L^p(0,1)},$$

where the inequality follows from Hölder's inequality. We have that $\|T_u\|_{(L^p(0,1))^*} \le \|u\|_{L^{p'}(0,1)}$.

For $\phi_u := |u|^{p'-2}u$, it is $|T_u\phi_u| = \|u\|^{p'}_{L^{p'}(0,1)}$ and $\|\phi_u\|_{L^p(0,1)} = \|u\|^{p'-1}_{L^{p'}(0,1)}$. Hence $\|T_u\|_{(L^p(0,1))^*} = \|u\|_{L^{p'}(0,1)}$.

(c) No such functional exists. It has been carefully shown at the exercise.

(d) Let $F \in Y^*$ be arbitrary. The mapping $F\colon Y \to \mathbb{K}$ can be restricted to X and $F|_X$ is now a linear mapping from X to $\mathbb{K}$. We need to show that it is bounded. We have

$$\|F\|_{X^*} = \sup_{\substack{x\in X\\ x\neq 0}} \frac{|F(x)|}{\|x\|_X} \le C \sup_{\substack{x\in X\\ x\neq 0}} \frac{|F(x)|}{\|x\|_Y} \le C \sup_{\substack{x\in Y\\ x\neq 0}} \frac{|F(x)|}{\|x\|_Y} = C\|F\|_{Y^*},$$

where we have used continuity $\|x\|_Y \le C\|x\|_X$ of the embedding in the first inequality and the inclusion $X \subset Y$ in the second inequality. Proposition 1 is thus proved.

We say X is continuously embedded in Y and we write $X \hookrightarrow Y$ if $X \subset Y$ and $\|x\|_Y \le C\|x\|_X$ for all $x \in X$ and a suitable $C > 0$. With this definition, Proposition 1 can be rephrased as follows: if $X \hookrightarrow Y$ then $Y^* \hookrightarrow X^*$.

Using Hölder's inequality we have

$$\|f\|^2_{L^2(0,1)} = \int_0^1 |f|^2 \le \big\||f|^2\big\|_{L^2(0,1)}\|1\|_{L^2(0,1)} = \|f\|^2_{L^4(0,1)},$$

which shows that $L^4(0,1) \hookrightarrow L^2(0,1)$. This implies, using Proposition 1, that $\big(L^2(0,1)\big)^* \hookrightarrow \big(L^4(0,1)\big)^*$. As the mapping $u \mapsto T_u$, as defined above, is an isometric isomorphism between $L^{p'}(0,1)$ and $(L^p(0,1))^*$, we obtain that $L^2(0,1) \hookrightarrow L^{4/3}(0,1)$.

(e) We claim that $\|x\|_X = \sup\limits_{\substack{\phi\in X^*\\ \|\phi\|_{X^*}\le 1}} |\phi(x)|$.

For a $\phi \in X^*$ with $\|\phi\|_{X^*} \le 1$ we have

$$|\phi(x)| \le \|\phi\|_{X^*}\|x\|_X \le \|x\|_X,$$

which shows that $\|x\|_X \ge \sup_{\|\phi\|_{X^*}\le 1} |\phi(x)|$. It remains to get the opposite inequality. If $x = 0$ this follows trivially, so assume the converse. By the Hahn–Banach theorem there exists $f \in X^*$ with $\|f\|_{X^*} = 1$ and $f(x) = \|x\|_X$. Hence

$$\|x\|_X = f(x) \le \sup_{\substack{\phi\in X^*\\ \|\phi\|_{X^*}\le 1}} |\phi(x)|.$$

[7] 4. **Convergences**

(a) Consider a bounded sequence of functions $f_n \in L^2((0,T))$. Give definition of $\{f_n\}_{n=1}^{\infty}$ converges to f weakly in $L^2((0,T))$. Show, as $n \to \infty$, that the following equivalence holds:

$$f_n \text{ converges to } f \text{ weakly in } L^2((0,T)) \qquad \Longleftrightarrow \qquad \int_0^b f_n(x)\,\mathrm{d}x \to \int_0^b f(x)\,\mathrm{d}x \quad \text{for every } b \in (0,T).$$

(b) Prove that any weakly converging sequence in a Hilbert space H is necessarily bounded.

(c) Consider in $L^2((0,\infty))$ the sequence of functions

$$f_n(x) = \begin{cases} n^{2/3} & \text{if } x \in [0,1/n] \\ 0 & \text{otherwise} \end{cases}$$

Is it true that $f_n \to 0$ weakly in $L^2((0,\infty))$?

Solution:

(a) We say that $\{f_n\}_{n=1}^{\infty}$ converges weakly to f in $L^2(0,T)$ if $G(f_n) \to G(f)$ as $n \to \infty$ for every $G \in (L^2(0,T))^*$. By the Riesz representation theorem this is equivalent to $(f_n, g) \to (f, g)$ as $n \to \infty$ for every $g \in L^2(0,T)$, where $(u,v) = \int_0^T u\,v$ is the inner product on $L^2(0,T)$.

If f_n converges weakly to f in $L^2(0,T)$, then, by definition and the Riesz representation theorem, $(f_n - f, g) \to 0$ for every $g \in L^2(0,T)$. Hence for arbitrary $b \in (0,T)$, $g := \chi_{(0,b)} \in L^2(0,T)$, the characteristic function of the interval $(0,b)$, we have $\int_0^b (f_n - f) \to 0$. As $b \in (0,T)$ was chosen arbitrarily, the right implication is clear.

Now assume that $\int_0^b (f_n - f) \to 0$ for every $b \in (0,T)$. Equivalently, $(f_n - f, \chi_{(0,b)}) \to 0$ for every $b \in (0,T)$. It is straightforward to show by continuity of $b \mapsto \|\chi_{(0,b)}\|_{L^2(0,T)}$ and Lebesgue's dominated convergence theorem that $(f_n - f, \chi_{(0,T)}) \to 0$ for every $b \in [0,T]$ (by taking the limits $b \to 0$ and $b \to T$). Consider the space of step functions $S = \operatorname{span}\{\chi_{(0,b)},\, b \in [0,T]\} \subset L^2(0,T)$, i.e., the space of finite linear combinations of characterisic functions. By linearity of the inner product, we thus have that $(f_n - f, h) \to 0$ for every $h \in S$. Now for an arbitrary $g \in L^2(0,T)$ and $\varepsilon > 0$, we can approximate g by a suitable $g_\varepsilon \in S$ such that $\|g - g_\varepsilon\|_{L^2(0,T)} < \varepsilon$. Hence,

$$|(f_n - f, g)| \le |(f_n - f, g - g_\varepsilon)| + |(f_n - f, g_\varepsilon)| \le \|f_n - f\|_{L^2(0,T)} \|g - g_\varepsilon\|_{L^2(0,T)} + |(f_n - f, g_\varepsilon)|,$$

where we have used Hölder's inequality. The second term on the right-hand side tends to zero as $n \to \infty$ for any but fixed $g_\varepsilon \in S$. As the sequence $\{\|f_n - f\|_{L^2(0,T)}\}$ is bounded as we will prove below, the first term can be made arbitrarily small independently of n by choosing suitable $g_\varepsilon \in S$. Thus $(f_n - f, g) \to 0$ for every $g \in L^2(0,T)$ and the proof is finished.

(b) Suppose that $y_n \rightharpoonup y$ weakly in H. Equivalently, $F(y_n - y) \to 0$ for every $F \in H^*$. Equivalently $G_n(F) \to 0$ for every $F \in H^*$, where $G_n = J(y_n - y) \in H^{**}$, the image of $y_n - y \in H$ under the canonical embedding J. Recall that J is an isometry, i.e., $\|G_n\|_{H^{**}} = \|y_n - y\|_H$. So it remains to show that $\{G_n\}_{n=1}^{\infty} \subset H^{**}$ is bounded in H^{**} to finish the proof. As the number sequence $G_n(F) \to 0$ for arbitrary fixed $F \in H^*$, $\{G_n(F)\}$ is bounded. By the uniform boundedness principle, $\{G_n\}$ is thus bounded in H^{**} and the proof is finished.

Note that this proof works even for arbitrary normed space H. We invoked the uniform boundedness principle for $\{G_n\} \subset H^{**}$, linear bounded operators from H^* to $\mathbb{R}$ (or $\mathbb{C}$). The space H^* is Banach even when H is only normed, so using the uniform boundedness principle is justified even in this case.

(c) As $\|f_n\|^2_{L^2(0,\infty)} = \int_0^{1/n} n^{4/3}\,\mathrm{d}x = n^{1/3}$, the sequence $\{f_n\}$ is unbounded in $L^2(0,\infty)$ and thus cannot converge weakly therein by virtue of the previous task.