$\boxed{1A}$ Je dána množina

$$\Omega = \{(x,y,z) : x^2+y^2+z^2 < 9,\ z < \sqrt{x^2+y^2},\ x>0\}$$

a nechť $\nu(x,y,z)$ je jednotkový vektor vnější normály určený ve s.v. bodech $(x,y,z) \in \partial\Omega$.

Spočtěte $\displaystyle\int_{\partial\Omega} \langle F,\nu\rangle \, dS$ pro vekt. pole

$$F(x,y,z) = (xy^2z,\ z,\ x^2+y^2).$$

Použijte Gauss-Ostrogradského větu.

<u>Řešení:</u> $\operatorname{div} F = y^2 z$

$$\int_{\partial\Omega} \langle F,\nu\rangle dS = \int_{\Omega} \operatorname{div} F \, dx^3 = \iiint\limits_{\left\{\substack{x^2+y^2+z^2<9 \\ z<\sqrt{x^2+y^2},\ x>0}\right\}} y^2 z \, dx\,dy\,dz = (*)$$

sfér. souř.: $\begin{cases} x = r\cos u \cos v \\ y = r \sin u \cos v \\ z = r \sin v \end{cases}$ $\quad r \in (0,3),\ u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ v \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$ $\quad \underline{\operatorname{Jac} = r^2 \cos v}$

$$(*) = \int_{-\pi/2}^{\pi/4} \int_{-\pi/2}^{\pi/2} \int_0^3 r^5 \sin^2 u \cos^3 v \sin v \, dr\,du\,dv$$

$$= \frac{3^6}{6} \cdot \frac{\pi}{2} \int_{-1}^{\sqrt{2}/2} (1-t^2)t\,dt = \frac{3^5}{4}\pi\left(-\frac{1}{16}\right) = \underline{\underline{-\frac{3^5}{2^6}\pi}}$$

③ **Řešení:**

(a) $\varphi(\alpha,t) = (\cos\alpha, \sin\alpha, t)$

(i) $\varphi \in C^\infty(U)$ (zřejmé)

(ii) $\operatorname{rank} D\varphi(\alpha,t) = 2$:

$$D\varphi(\alpha,t) = \begin{pmatrix} -\sin\alpha, & 0 \\ \cos\alpha, & 0 \\ 0, & 1 \end{pmatrix}, \quad (D\varphi)^T(D\varphi) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

(iii) φ lze rozšířit na $\mathbb{R}^2$ jako C^∞ zobrazení, φ je prosté na U ($\cos\alpha = \cos\alpha'$, $\sin\alpha = \sin\alpha' \Rightarrow \alpha = \alpha'$)

$\varphi(U) \cap \varphi(\overline{U} \setminus U) = \emptyset$

$\Rightarrow \varphi$ je mapa na U (Věta 1.2 z předn.)

- S je válcová plocha $\{x^2+y^2=1,\ 0<z<1\}$ bez úsečky $\{x=-1, y=0\}$
- T je torus $(R=3, r=1)$ bez 2 kružnic

(b) $\Phi: S \to \mathbb{R}^3$ def. na ploše S

$\varphi: U \to S$ mapa, $\Phi\circ\varphi$ diferenc.

$\Rightarrow \Phi$ diferenc. na S (Def. 1.5 z předn.)

Podobně $\Phi^{-1}: T \to \mathbb{R}^3$ def. na ploše T

$\Phi^{-1}\circ\psi = \varphi$ diferenc., ψ mapa S

$\Rightarrow \Phi^{-1}$ diferenc. na S

(c) $p = \varphi(\alpha, t)$

$$T_p S = \mathrm{Lin}\{\underbrace{\frac{\partial \varphi}{\partial \alpha}(\alpha,t), \frac{\partial \varphi}{\partial t}(\alpha,t)}_{B_1}\}$$

$$= \mathrm{Lin}\left\{\begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$T_{\Phi(p)} T = \mathrm{Lin}\{\underbrace{\frac{\partial \psi}{\partial \alpha}(\alpha,t), \frac{\partial \psi}{\partial t}(\alpha,t)}_{B_2}\}$$

$$= \mathrm{Lin}\left\{\begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin(2\pi t)\cos\alpha \\ -\sin(2\pi t)\sin\alpha \\ \cos(2\pi t) \end{pmatrix}\right\}$$

(d) $$D\Phi(p) = D(\overbrace{\Phi \circ p}^{\psi})(\alpha,t) \circ (d\varphi(\alpha,t))^{-1}$$

(dle definice)

$$= d\psi(\alpha,t) \circ d\varphi(\alpha,t)^{-1}$$

$d\varphi(\alpha,t)$: $e_1 \longmapsto \frac{\partial \varphi}{\partial \alpha}$, $e_2 \longmapsto \frac{\partial \varphi}{\partial t}$ | $d\psi(\alpha,t)$: $e_1 \longmapsto \frac{\partial \psi}{\partial \alpha}$, $e_2 \longmapsto \frac{\partial \psi}{\partial t}$

$$\underbrace{d\psi(\alpha,t) \circ d\varphi(\alpha,t)^{-1}}_{T_p S \to T_{\Phi(p)} T} : \left[\begin{array}{l} \frac{\partial \varphi}{\partial \alpha} \longmapsto \frac{\partial \psi}{\partial \alpha} \\ \frac{\partial \varphi}{\partial t} \longmapsto \frac{\partial \psi}{\partial t} \end{array}\right]$$

má matici $\begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix}$ vzhledem k bázím B_1, B_2