Algebra I

30.01.2026

2.5 hours

Grading system: 0-2 pt = “4”, 3-5 pt = “3”, 6-8 pt = “2”, 9-11 pt = “1”

Exercise 1 (1pt) Let G be a finite set with a binary operation $*\colon G \times G \to G$ such that:

- $*$ is associative,
- $*$ is commutative,
- given $a, b, c \in G$, $b * a = c * a$ implies $b = c$.

Show that $(G, *)$ is a group. Is it true when G is infinite?

Exercise 2 (1pt) Show that $\mathbf{S}_n = \langle (1\,2\,\ldots\,n-1), (1\,2\,\ldots\,n) \rangle$.

Exercise 3 (1pt) Compute $18^{2026} \pmod{80}$.

Exercise 4 (1pt+1pt) An element $x \neq 0$ in a ring $\mathbf{A}$ is *nilpotent* if $x^k = 0$ for some $k \in \mathbb{N}$.

(1) Show that if x is nilpotent, then $1 - x$ has a multiplicative inverse in $\mathbf{A}$.

(2) Show that the ring $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ contains a nilpotent element if and only if $n^2 | m$, for some $n > 1$.

Exercise 5 (2pt) Show that, in a ring with unity, the *commutativity of* + axiom can be deduced from the other axioms.

Exercise 6 (1pt+1pt) Let $p \in \mathbb{N}$ be a prime number.

(1) Find all the roots of the polynomial $h(x) = x^{p-1} - 1$ in $\mathbb{Z}_p$.

(2) Using item 1, show that $(p-1)! \equiv -1 \pmod{p}$.

Hint: Let $g(x) = (x-1)(x-2)\ldots(x-(p-1))$ and $f(x) = g(x) - h(x)$. What are the roots of $f(x)$ in $\mathbb{Z}_p$? What is the degree of $f(x)$?

Exercise 7 (2pt) For an edge-colored graph G, an *automorphism* of G is a bijection from the set of vertices $V(G)$ to itself which preserves both edges and their colors. The set of automorphisms of G forms a group that acts on $V(G)$ with permutations.

In how many ways can we color the edges of K_4 (complete graph on 4 vertices) with n colors up to automorphism?